On a delay implicit functional integro-differential equation

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ABSTRACT: In this work, we use the De Blasi measure of noncompactness and Darbo fixed point Theorem to study the existence of solutions for an initial value problem of a delay implicit functional integro-differential equation. The sufficient condition for the uniqueness of the solution will be given. The continuous dependence of the unique solution on some data will be studied.

1. INTRODUCTION

The study of implicit differential and integral equations has received much attention over the last 30 years or so. For papers studying such kind of problems (see [15,16,17,18]) and the references therein.

For the theoretical results concerning the existence of solutions, in the classes of continuous or integrable functions, you can see Banas’s [18–21]. Each of these monographs contains some existence results, and the main objective is to present a technique to obtain some results concerning various integral equations.

Here we are concerning with the initial value problem of the delay implicit functional integro-differential equation

\[
\frac{dx}{dt} = f(t, \frac{dx}{dt}, \int_0^t g(s, x(s))ds), \ a.e \ t \in (0,1]
\]

with the initial data

\[x(0) = x_0, \] (2)

Let \( \frac{dx}{dt} = y(t) \), then the solution of the problem (1)-(2) can be given by

\[x(t) = x_0 + \int_0^t y(s)ds, \] (3)

where \( y \) is the solution of the functional integral equation

\[y(t) = f(t, y(t), \int_0^t g(s, x(s))ds). \] (4)

We study the existence of nondecreasing solutions \( y \in L_1[0,1] \) of the integral equation (4) will be studied by the De Blasi measure of noncompactness [1], Hausdorff measure of noncompactness \( \chi[2] \) and Darbo fixed point Theorem [4]. The sufficient condition for the uniqueness of the solution will be given. The continuous dependence of the unique solution on the initial data \( x_0 \) and on the functions \( g \) and \( \phi \) will be studied.

Consequently, the existence of absolutely continuous solution \( x \in AC[0,1] \), the unique solution and the continuous dependence of the unique solution on the initial data \( x_0 \) and on the functions \( g \) and \( \phi \) of the problem (1)-(2) will be studied.

We arrange our article just like that: Section 2 the Some properties and theories used. In Section 3, contains the solvability for the existence of the solutions \( x \in AC[0,1] \). Moreover, we study the unique solution \( x \in AC[0,1] \) of the problem (1)-(2) and its continuous dependence on the initial data \( x_0 \) and on the functions \( g \) and \( \phi \) of the problem (1)-(2).

Some examples in Section 4.

2. Preliminaries

Let \( L_1 = L_1(I) \) be the class of Lebesgue integrable functions on the interval \( I = [0,1] \), with the standard norm

\[\|x\|_1 = \int_0^1 |x(t)|dt.\]

Theorem 1. Let \( x \) be a bounded subset of \( L_1 \). Assume that there is a family of measurable subsets \( \{\Omega_c\} \) \( 0 \leq c \leq b - a \) of the interval \( (a,b) \) such that \( \text{mes} \Omega_c = c \). If for every \( c \in [0, b - a] \) and for every

\[x \in X, \quad x(t_1) \leq x(t_2), \quad (t_1 \in \Omega_c, t_2 \in \Omega_c)\]

then, the set \( X \) is compact in measure.
3. Continuous dependence

Consider now the initial value problem (1)-(2) under the following assumptions:

(i) \( \phi : I \to I \), \( \phi(t) \leq t \) is continuous and increasing.
(ii) \( f : I \times R \times R \to R^+ \) is a Carathéodory function which is measurable \( t \in I \) for any \( x, y \in R \times R \) and continuous in \( x, y \in R \times R \) for all \( t \in I \) and there exists a measurable and bounded function \( m_1 : I \to R \) and a positive constant \( b_1 \) such that

\[
|f(t,x,y)| \leq m_1 + b_1|x| + |y|.
\]

Moreover, \( f \) is nondecreasing for every nondecreasing \( x, y \) i.e. \( \forall t \in I \), \( t_1 \leq t_2 \) and for all \( x(t_1) \leq x(t_2) \) and \( y(t_1) \leq y(t_2) \) implies

\[
 f(t_1, x(t_1), y(t_1)) \leq f(t_2, x(t_2), y(t_2)).
\]

(iii) \( g : I \times R \times R \to R^+ \) is a Carathéodory function which is measurable in \( t \in I \) for any \( x \in R \) and continuous in \( x \in R \) for all \( t \in I \). Moreover, there exist an integrable function \( m_2 : I \to R \) and a positive constant \( b_2 \) such that

\[
g(t,x,y) \leq m_2 + b_2|x|.
\]

(iv) \( b_1 + b_1b_2 < 1 \).

Now, the following lemma can be proved.

**Lemma 1.** The problem (1)-(2) is equivalent to the integral equation

\[
x(t) = x_0 + \int_0^t y(s)ds,
\]

where

\[
y(t) = f \left(t, y(t), \int_0^t g(s,x_0 + \int_0^s y(\theta)d\theta) \right).
\]

Now, we have the following existence theorem.

**Theorem 2.** Let the assumptions (i)-(iv) be satisfied, then the equation (4) has at least one nondecreasing solution \( x \in AC(I) \).

**Proof.** Let \( Q_r \) be the closed ball of all nondecreasing function on \( I \)

\[
Q_r = \{ y \in L_1(I) : \| y \| \leq r \}, \quad r = \frac{\|m_2\|1 + b_1 + b_2|x_0| + \|m_1\|1}{1-(b_1+b_1b_2)},
\]

and the supper position operator \( F \)

\[
F(y) = f \left(t, y(t), \int_0^t g(s,x_0 + \int_0^s y(\theta)d\theta) \right).
\]

Then, we deduce that \( F \) transforms the nondecreasing functions into functions of the same type.

From our assumptions and by Theorem 1, \( Q_r \) is compact in measure.

Now, let \( y \in Q_r \), then

\[
|F(y)(t)| = |f \left(t, y(t), \int_0^t g(s,x_0 + \int_0^s y(\theta)d\theta) \right)|
\]

\[
\leq |m_1(t)| + b_1 \left( |y(t)| + \int_0^t g(s,x_0 + \int_0^s y(\theta)d\theta) \right) 
\]

\[
\leq |m_1(t)| + b_1 |y(t)| + b_1 \int_0^t \left( m_2(s) + b_1b_2|x_0| \right) ds + b_1b_2 \int_0^t \int_0^s y(\theta)d\theta ds
\]

\[
\leq |m_1(t)| + b_1 |y(t)| + b_1 \int_0^t m_2(s) ds + b_1b_2|x_0| + b_1b_2 \int_0^t \int_0^s y(\theta)d\theta ds
\]

and

\[
\int_0^1 \int_0^t |y(s)| ds < \infty,
\]

Then applying the De Blasi measure of noncompactness [1,2,4,8].

\[
\beta(F(t)) \leq \beta (y(t))(b_1 + b_1b_2)
\]

and

\[
\beta(F(I)) \leq \beta (\Omega(t))(b_1 + b_1b_2)
\]

Then implies

\[
\chi(F) \leq \chi(\Omega)
\]

Where \( \chi \) is the Hausdorff measure of noncompactness [1,2,4,8]. Since \( b_1 + b_1b_2 < 1 \), from Darbo fixed point Theorem [4] \( F \) is a contraction with regard to the measure of noncompactness \( \chi \) [4] and has at least one fixed point \( y \in Q_r \). Then there exist at least one solution \( y \in L_1(I) \) of equation (4). Consequently there
exists at least one solution 
\( x \in AC(I) \) of the problem (1)-(2).

**3.1 Uniqueness of the solution**

Now, consider the following assumptions:

1. **(i)** \( f : I \times R \times R \rightarrow R \) is measurable in \( t \in I \) \( \forall x, y \in R \) and satisfies Lipschitz condition,
   \[
   |f(t, x, y) - f(t, x_1, y_1)| \leq b_1(|x - x_1| + |y - y_1|), \quad t \in I \quad x, y, x_1, y_1 \in R
   \]
   and \( \int_0^t |f(t, 0, 0)| dt \) exists. Moreover, \( f \) is nondecreasing for every nondecreasing \( x, y \) i.e. for almost all \( t_1, t_2 \in I \) such that \( t_1 \leq t_2 \) and for all \( x(t_1) \leq x(t_2) \) and \( y(t_1) \leq y(t_2) \) impales \( f(t_2, x(t_2), y(t_1)) \leq f(t_2, x(t_2), y(t_2)). \)

2. **(ii)** \( g : I \times R \rightarrow R \) is measurable in \( t \in I \) and satisfies Lipschitz condition,
   \[
   |g(t, x) - g(t, y)| \leq b_2|x - y|, \quad t \in I, \quad x, y \in R.
   \]
   From the assumption (ii) we have
   \[
   |f(t, x, y) - f(t, 0, 0)| \leq b_1(|x| + |y|) \quad \text{and} \quad |f(t, x, y)| \leq m_1 + b_1(|x| + |y|).
   \]
   Also, from the assumption (iii) we get
   \[
   |g(t, x) - g(t, 0)| \leq b_2|x|,
   \]
   and
   \[
   |g(t, x)| \leq m_2 + b_2|x|.
   \]
   So, we have proved the following Lemma.

**Lemma 2.** The assumptions (ii) and (iii) implies the assumptions (ii) and (iii) respectively.

**Theorem 3.** Let the assumptions (i), (ii), (iii) and (iv) be satisfied. If

\[
|b_1 + b_2| < 1,
\]

Then the solution of the problem (1)-(2) is unique.

**Proof.** Form Lemma 2 the assumptions of Theorem 1 are satisfied and the solution of integral equation (4) exists. Let \( y_1, y_2 \) be two solutions in \( C_0 \) of the integral equation (4), then

\[
|y_2(t) - y_1(t)| = \left| f\left(t, y_2(t), \int_0^t \phi(t, s, y_1(s) + \int_0^s \phi(s, z_1(z) d\theta) ds\right)\right|
\]

\[
- \left| f\left(t, y_1(t), \int_0^t \phi(t, s, y_1(s) + \int_0^s \phi(s, z_1(z) d\theta) ds\right)\right|
\]

\[
\leq \left| f\left(t, y_2(t), \int_0^t \phi(t, s, y_1(s) + \int_0^s \phi(s, z_1(z) d\theta) ds\right)\right|
\]

\[
- \left| f\left(t, y_1(t), \int_0^t \phi(t, s, y_1(s) + \int_0^s \phi(s, z_1(z) d\theta) ds\right)\right|
\]

\[
\leq b_1 \int_0^1 |y_2(t) - y_1(t)| dt \leq b_1 \int_0^t |y_2(t) - y_1(t)| dt.
\]

Hence

\[
\|y_2 - y_1\|_1 \leq b_1 \|y_2 - y_1\|_1 + b_1 \|y_2 - y_1\|_1.
\]

Then

\[
\|y_2 - y_1\|_1 \leq b_1 \|y_2 - y_1\|_1 + b_1 \|y_2 - y_1\|_1.
\]

And

\[
\|y_2 - y_1\|_1 \leq \|f(y_2) - f(y_1)\|_1 \leq b_1 \|y_2 - y_1\|_1 + b_1 \|y_2 - y_1\|_1.
\]

Hence

\[
\|y_2 - y_1\|_1 \leq b_1 \|y_2 - y_1\|_1 + b_1 \|y_2 - y_1\|_1.
\]

Then

\[
\|y_2 - y_1\|_1 \leq \frac{b_1}{1-b_1} \|y_2 - y_1\|_1,
\]

and

\[
\|y_2 - y_1\|_1 \leq \frac{b_1}{1-b_1} \|y_2 - y_1\|_1.
\]

Hence

\[
\|y_2 - y_1\|_1 \leq \frac{b_1}{1-b_1} \|y_2 - y_1\|_1.
\]

Then

\[
\|y_2 - y_1\|_1 \leq \frac{b_1}{1-b_1} \|y_2 - y_1\|_1.
\]

And

\[
\|x - x^*\|_c \leq \frac{\|y_2 - y_1\|_1}{\|y_2 - y_1\|_1} = \epsilon.
\]

**Definition.** The solution of the initial value problem (1)-(2) depends continuously on the parameter \( x_0 \), if

\[
\forall \epsilon > 0, \quad \exists \delta(\epsilon) > 0, \quad t \|x_0 - x_0^*\| < \delta \Rightarrow \|x - x^*\| < \epsilon.
\]

Where \( x^* \)

\[
x^*(t) = x_0^* + \int_0^t \gamma(s) ds.
\]

**Theorem 4.** Let the assumptions of Theorem 3 be satisfied, then the unique solution of the problem (1)-(2) depends continuously on the parameter \( x_0 \).

**Proof.** Let \( \delta > 0 \) be given such that \( |x_0 - x_0^*| \leq \delta \) and let \( x_0^* \) be the solution

of (1)-(2) corresponding to initial value \( x_0^* \), then

\[
\|x(t) - x^*(t)\| = |x_0 + \int_0^t \gamma(s) ds - x_0^* - \int_0^t \gamma^*(s) ds|
\]

\[
\leq |x_0 - x_0^*| + \int_0^t \gamma(s) - \gamma^*(s) ds \leq \delta + \|\gamma - \gamma^*\|_1
\]

But

\[
\|\gamma(t) - \gamma^*(t)\| = |f(t, y(t), \int_0^t \phi(t, s, x_0 + \int_0^s \gamma^*(z) d\theta) ds|
\]

\[
- f(t, y^*(t), \int_0^t \phi(t, s, x_0^* + \int_0^s \gamma^*(z) d\theta) ds|
\]

\[
\leq |f(t, y(t), \int_0^t \phi(t, s, x_0 + \int_0^s \gamma^*(z) d\theta) ds|
\]

\[
- f(t, y^*(t), \int_0^t \phi(t, s, x_0^* + \int_0^s \gamma^*(z) d\theta) ds|
\]

\[
\leq b_1 \int_0^1 |y(t) - y^*(t)| dt \leq b_1 \int_0^1 |y(t) - y^*(t)| dt + b_1 \int_0^1 |y(t) - y^*(t)| dt
\]

\[
\leq b_1 b_2 \delta + b_1 b_2 \|y - y^*\|_1 + b_1 \|y - y^*\|_1.
\]

Then

\[
\int_0^1 |y(t) - y^*(t)| dt \leq b_1 b_2 \delta + b_1 b_2 \|y - y^*\|_1 + b_1 \|y - y^*\|_1.
\]

Hence

\[
\|y - y^*\|_1 \leq \delta b_1 b_2 + b_1 b_2 \|y - y^*\|_1 + b_1 \|y - y^*\|_1.
\]

Then

\[
\|y - y^*\|_1 \leq \frac{\delta b_1 b_2}{1-(b_1 + b_2)} = \epsilon_1
\]

and

\[
\|x - x^*\|_c \leq \delta + \epsilon_1 = \epsilon.
\]
Definition 2. The solution \( x \) of the initial value problem (1)-(2) depends continuously on the function \( g \), if
\[
\forall \varepsilon > 0, \exists \delta (\varepsilon) > 0 \text{ s.t. } |g(t,x) - g*(t,x)| < \delta \Rightarrow \| x - x^* \| < \varepsilon.
\]

Theorem 3. Let the assumptions of Theorem 3 be satisfied, then the unique solution of the problem (1)-(2) depends continuously on the function \( g \).

Proof. Let \( \delta > 0 \) be given such that \( |g(t,x(t)) - g*(t,x(t))| \leq \delta \) and let \( x^* \) be the solution of (1)-(2) corresponding to \( g*(t,x) \), then
\[
|x(t) - x^*(t)| = |x_0 + \int_0^t g(s)ds| - \int_0^t y^*(s)ds| \\
\leq |\int_0^t |y(s) - y^*(s)|ds| \leq \| y - y^* \|_1
\]

But
\[
|y(t) - y^*(t)| = |f(t,y(t),\int_0^t g(s,x_0 + \int_0^s y^*(\theta)d\theta)ds) - f(t,y^*(t),\int_0^t g(s,x_0 + \int_0^s y^*(\theta)d\theta)ds)|
\]

Then
\[
\int_0^t |y(t) - y^*(t)| dt \\leq \int_0^t \int_0^s \int_0^{y^*(\theta)}ds\cdot d\theta = b_1b_2 \int_0^t \int_0^s \int_0^{y^*(\theta)}ds\cdot d\theta = b_1b_2 \int_0^t \int_0^s \int_0^{y^*(\theta)}ds\cdot d\theta
\]

Hence
\[
\| y - y^* \|_1 \leq \int_0^t \int_0^s \int_0^{y^*(\theta)}ds\cdot d\theta = b_1b_2 \int_0^t \int_0^s \int_0^{y^*(\theta)}ds\cdot d\theta = b_1b_2 \int_0^t \int_0^s \int_0^{y^*(\theta)}ds\cdot d\theta
\]

Then
\[
\| y - y^* \|_1 \leq \frac{\delta b_1}{(\delta b_1 + b_2b_2)} = \varepsilon
\]

and
\[
\| x - x^* \|_C \leq \varepsilon.
\]

Definition 3. The solution \( x \) of the initial value problem (1)-(2) depends continuously on the function \( \phi \), if
\[
\forall \varepsilon > 0, \exists \delta (\varepsilon) > 0 \text{ s.t. } |\phi(t) - \phi^*(t)| < \delta \Rightarrow \| x - x^* \| < \varepsilon.
\]

Theorem 6. Let the assumptions of Theorem 3 be satisfied, then the unique solution of the problem (1)-(2) depends continuously on the delay function \( \phi \).

Proof. Let \( \delta > 0 \) be given such that \( |\phi(t) - \phi^*(t)| \leq \delta \) and let \( x^* \) be the solution of (1)-(2) corresponding to \( \phi^*(t) \), then
\[
|x(t) - x^*(t)| = |x_0 + \int_0^t y^*(s)ds - x_0 - \int_0^t y^*(s)ds| \\
\leq \int_0^t |y(s) - y^*(s)|ds| \leq \| y - y^* \|_1
\]

But
\[
|y(t) - y^*(t)| = |f(t,y(t),\int_0^t g(s,x_0 + \int_0^s y^*(\theta)d\theta)ds) - f(t,y^*(t),\int_0^t g(s,x_0 + \int_0^s y^*(\theta)d\theta)ds)|
\]

\[
\leq \int_0^t |y(s) - y^*(s)|ds| \leq \| y - y^* \|_1
\]

\[
\int_0^t |\phi(t) - \phi^*(t)| |g(s,x_0 + \int_0^s y^*(\theta)d\theta)ds) - g(s,x_0 + \int_0^s y^*(\theta)d\theta)ds)|
\]

\[
\leq |f(t,y(t),\int_0^t g(s,x_0 + \int_0^s y^*(\theta)d\theta)ds) - f(t,y^*(t),\int_0^t g(s,x_0 + \int_0^s y^*(\theta)d\theta)ds)|
\]

\[
\leq b_1 \int_0^t |g(s,x_0 + \int_0^s y^*(\theta)d\theta)ds) - g(s,x_0 + \int_0^s y^*(\theta)d\theta)ds)|
\]

Then
\[
\int_0^t |y(t) - y^*(t)| dt \leq b_1b_2 \int_0^t |y(t) - y^*(t)| dt + b_1b_2 \int_0^t \int_0^s \int_0^{y^*(\theta)}ds\cdot d\theta = b_1b_2 \int_0^t \int_0^s \int_0^{y^*(\theta)}ds\cdot d\theta
\]

Hence
\[
\| y - y^* \|_1 \leq \int_0^t \int_0^s \int_0^{y^*(\theta)}ds\cdot d\theta = b_1b_2 \int_0^t \int_0^s \int_0^{y^*(\theta)}ds\cdot d\theta
\]

Then
\[
\| x - x^* \|_C \leq \varepsilon.
\]
$r = 0.45$. Then the initial value problem (5)-(6) has at least one solution.

4. Conclusions

In this paper, we have studied a delay implicit functional integro-differential equation. We have proved the existence of solutions for an initial value problem of a delay implicit functional integro-differential equation. Then we have established the sufficient conditions for the uniqueness of solution and continuous dependence of solution on some initial data and the functions $g$, $\phi$ are studied. An example is given to illustrate our results.

References


