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Article

Solvability of an initial-value problem of non-linear implicit fractal differential equation

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ABSTRACT: In this paper we study the initial-value problem of the fractal differential equation

$$\frac{dx(t)}{dt^{\beta}} = f\left(t, \frac{dx(t)}{dt^{\alpha}}\right), \quad a.e. \quad t \in (0,T], \quad x(0) = x_0.$$

We discuss the existence of at least one solution $x \in \mathrm{AC}[0,\mathrm{T}]$. The Uniqueness of the solution will be proved. The continuous dependence on the initial data x_0 and on the function f will be analysed. Also, the Hyers - Ulam stability of the problem will be established. Finally, some examples will be given to verify our results.

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1. INTRODCTION

Implicit fractal differential equations represent a fascinating area of study that combines fractal geometry and differential equations. These equations involve fractal-like structures and demonstrate intricate and self-similar patterns (see [2, 11]). They are characterized by their non-linearity and often exhibit complex behaviours, such as chaos and self-replication (see [13, 14]). Differential equations and fractal differential equations have applications in various fields, including physics, biology, and finance, and have garnered significant interest due to their ability to model complex systems with remarkable precision (see [1, 4, 6, 10, 12]).

In this paper we will focus on this initial-value problem of non-linear implicit fractal differential equation.

$$\frac{dx(t)}{dt^{\beta}} = f\left(t, \frac{dx(t)}{dt^{\alpha}}\right), \quad a.e. \ t \in (0,T], \quad x(0) = x_0, \qquad (1)$$

where $\beta, \alpha \in (0,1)$, and $\beta > \alpha$.

2. Existence of solution

Let $I = [0,T]$ and suppose the following conditions:

(i) $f : I \times R \to R$ is measurable in $t \in I$ for any $x \in R$ and continuous in $x \in R$ for almost all $t \in I$.

(ii) There exist a function $a \in L_1(I)$ such that $\int_0^T s^{\beta-1} |a(s)| ds \le M$ and a positive constant b such that

$$|f(t,x)| \le a(t) + b|x|, \qquad b > 0.$$

(iii) $b \, \frac{\beta}{\alpha} T^{\beta-\alpha} < 1$

Lemma 1. The solution of the problem (1), if it exists, then it can be given by the integral equation.

$$x(t) = x_0 + \int_0^t y(s)\, ds, \qquad t \in (0,T], \qquad (2)$$

where y is the solution of the functional equation.

$$y(t) = \beta\, t^{\beta-1} f\left(t, \frac{1}{\alpha} t^{1-\alpha} y(t)\right). \qquad (3)$$

Proof. Let x be a solution of (1), then we have

$$\frac{dx(t)}{dt}\frac{dt}{dt^{\beta}} = f\left(t, \frac{dx(t)}{dt}\frac{dt}{dt^{\alpha}}\right),$$

$$\frac{dx(t)}{dt}\frac{1}{\beta\, t^{\beta-1}} = f\left(t, \frac{1}{\alpha\, t^{\alpha-1}}\frac{dx(t)}{dt}\right),$$

$$\frac{dx(t)}{dt} = \beta\, t^{\beta-1} f\left(t, \frac{1}{\alpha\, t^{\alpha-1}}\frac{dx(t)}{dt}\right).$$

Let $y(t) = \frac{dx(t)}{dt}$, we obtain

$$x(t) = x_0 + \int_0^t y(s)\, ds$$

and

$$y(t) = \beta\, t^{\beta-1} f\left(t, \frac{1}{\alpha\, t^{\alpha-1}}\, y(t)\right).$$

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Let y be a solution of (3), then from (2)

$$x(t) = x_0 + \beta \int_0^t s^{\beta-1} f(s, \frac{1}{\alpha} s^{1-\alpha} \; y(s)) \, ds$$
$$= x_0 + \beta \int_0^t s^{\beta-1} f(s, \frac{1}{\alpha} s^{1-\alpha} \; \frac{dx(s)}{ds}) \, ds.$$

Thus

$$\frac{dx(t)}{dt^\beta} = \beta \frac{d}{dt^\beta} \int_0^t s^{\beta-1} f(s, \frac{1}{\alpha} s^{1-\alpha} \; \frac{dx(s)}{ds}) \, ds$$
$$= \beta \frac{dt}{dt^\beta} \frac{d}{dt} \int_0^t s^{\beta-1} f(s, \frac{1}{\alpha} s^{1-\alpha} \; \frac{dx(s)}{ds}) \, ds$$
$$= \beta \frac{1}{\beta t^{\beta-1}} t^{\beta-1} f(t, \frac{1}{\alpha} t^{1-\alpha} \; \frac{dx(t)}{dt})$$
$$= f(t, \frac{dx(t)}{dt^\alpha}), \quad a.e. \quad t \in (0, T].$$

Using (2), we get $x(0) = x_0$. Which completes the proof.

Now, we have the following existences theorem.

Theorem 1. Let the assumptions (i)-(iii) be satisfied, then the functional equation (3) has at least one integrable solution $y \in L_1(I)$. Consequently, the initial-value problem (1) has at least one solution x ∈ AC(I).

Proof. Let Q_r be the closed ball

$$Q_r = \{y \in R : \|y\|_{L_1(I)} \leq r\}, \qquad r = \frac{\beta M}{1 - b\frac{\beta}{\alpha} T^{\beta-\alpha}}$$

and the operator F is defined by

$$Fy(t) = \beta t^{\beta-1} f(t, \frac{1}{\alpha} t^{1-\alpha} y(t)).$$

Now, let $y \in Q_r$, then

$$|Fy(t)| = | \beta t^{\beta-1} f(t, \frac{1}{\alpha} t^{1-\alpha} y(t)) |$$
$$\leq \beta t^{\beta-1} | f(t, \frac{1}{\alpha} t^{1-\alpha} y(t)) |$$
$$\leq \beta t^{\beta-1} (|a(t)| + b | \frac{1}{\alpha} t^{1-\alpha} y(t) |)$$
$$\leq \beta t^{\beta-1} |a(t)| + b \frac{\beta}{\alpha} T^{\beta-\alpha} |y(t)|$$

and

$$\int_0^T |Fy(s)| \, ds \leq \beta \int_0^T s^{\beta-1} |a(s)| \, ds + b \frac{\beta}{\alpha} T^{\beta-\alpha} \int_0^T |y(s)| \, ds$$
$$\leq \beta M + b \frac{\beta}{\alpha} T^{\beta-\alpha} \|y\|_{L_1}$$
$$\leq \beta M + b \frac{\beta}{\alpha} T^{\beta-\alpha} r,$$

Thus

$$\|Fy\| \leq \beta M + b \frac{\beta}{\alpha} T^{\beta-\alpha} r = r.$$

This proves that $F: Q_r \to Q_r$ and the class of function $\{Fy\}$ is uniformly bounded in Q_r.

Now, let $y \in Q_r$, then

$$\int_0^T |(Fy(s))_h - (Fy(s))| ds = \int_0^T | \frac{1}{h} \int_t^{t+h} Fy(\theta) d\theta - Fy(s) | ds$$
$$= \int_0^T \frac{1}{h} | \int_t^{t+h} (Fy(\theta) - Fy(s)) d\theta | \, ds$$
$$\leq \int_0^T \frac{1}{h} \int_t^{t+h} | Fy(\theta) - Fy(s) | d\theta \, ds,$$

Thus

$$\|(Fy)_h - (Fy)\|_{L_1} \leq \int_0^T \frac{1}{h} \int_t^{t+h} | \beta \theta^{\beta-1} f(\theta, \frac{1}{\alpha} \theta^{1-\alpha} y(\theta))$$
$$- \beta s^{\beta-1} f(s, \frac{1}{\alpha} s^{1-\alpha} y(s)) | \, d\theta \, ds$$

Since $Fy = \beta t^{\beta-1} f(t, \frac{1}{\alpha} t^{1-\alpha} y) \in Q_r \subset L_1(I)$, it follows that

$$\int_0^T \frac{1}{h} \int_t^{t+h} | \beta \theta^{\beta-1} f(\theta, \frac{1}{\alpha} \theta^{1-\alpha} y(\theta)) -$$
$$\beta s^{\beta-1} f(s, \frac{1}{\alpha} s^{1-\alpha} y(s)) | \, d\theta \, ds \to 0, \; as \; h \to 0.$$

Then $(Fy)_h \to (Fy)$ uniformly in $L_1(I)$. Thus the class $\{Fy\}$, $y \in Q_r$ is relatively compact [9]. Hence F is compact operator [3].

Now, let $\{y_n\} \subset Q_r$, and $y_n \to y$, then

$$Fy_n(t) = \beta \; t^{\beta-1} f(t, \frac{1}{\alpha} t^{1-\alpha} y_n(t))$$

and

$$\lim_{n\to\infty} Fy_n(t) = \beta \; t^{\beta-1} \lim_{n\to\infty} f(t, \frac{1}{\alpha} t^{1-\alpha} y_n(t))$$
$$= \beta \; t^{\beta-1} f(t, \frac{1}{\alpha} t^{1-\alpha} \lim_{n\to\infty} y_n(t))$$
$$= \beta \; t^{\beta-1} f(t, \frac{1}{\alpha} t^{1-\alpha} y(t))$$
$$= Fy(t).$$

This means that $Fy_n(t) \to Fy(t)$. Hence the operator F is continuous.

Now, by Schauder fixed point Theorem [8] there exists at least one solution $y \in Q_r \subset L_1(I)$ of the functional equation (3). Consequently, there exists at least one solution $x \in$ AC(I) of the problem (1).

3. Uniqueness of the solution

Now, consider the following assumption:

$(i)^*$ f : I × R → R is measurable in t ∈ I for every $x \in$ R and satisfies the Lipschitz condition

$$|f(t,x) - f(t,y)| \leq b \, |x - y|, \qquad b > 0, \tag{4}$$

where $|f(t,0)| = a(t)$ is bounded.

Remark 1. From (4) we obtain

$$|f(t,x)| - |f(t,0)| \leq |f(t,x) - f(t,0)| \leq b \, |x|$$
$$|f(t,x)| \leq a(t) + b \, |x|, \qquad b > 0.$$

Theorem 2. Let the assumption $(i)^*$ be satisfied. If $b \frac{\beta}{\alpha} T^{\beta-\alpha} < 1$, then the solution y ∈ L_1(I) of the functional equation (3) is unique. Consequently, the solution $x \in$ AC(I) of the problem (1) is unique.

Proof. From remark 1, all assumptions of Theorem 1 are satisfied and the solution of the functional equation (3) exists.

Now, let y, z be two solutions of equation (3), then

$$|y(t) - z(t)| =$$
$$| \beta t^{\beta-1} f(t, \frac{1}{\alpha} t^{1-\alpha} \; y(t)) - \beta t^{\beta-1} f(t, \frac{1}{\alpha} t^{1-\alpha} \; z(t)) \; |$$
$$= \beta t^{\beta-1} | f(t, \frac{1}{\alpha} t^{1-\alpha} \; y(t)) - f(t, \frac{1}{\alpha} t^{1-\alpha} \; z(t)) |$$
$$\leq \beta t^{\beta-1} b \; | \frac{1}{\alpha} t^{1-\alpha} \; y(t) - \frac{1}{\alpha} t^{1-\alpha} \; z(t) |$$
$$\leq b \frac{\beta}{\alpha} T^{\beta-\alpha} |y(t) - z(t)|$$

and

$$\int_0^T |y(s) - z(s)| \, ds \leq b \frac{\beta}{\alpha} T^{\beta-\alpha} \int_0^T |y(s) - z(s)| \, ds$$
$$\|y - z\|_{L_1} \leq b \frac{\beta}{\alpha} T^{\beta-\alpha} \|y - z\|_{L_1}.$$

Hence,

$$\|y-z\|_{L_1}\,(1-\mathrm{b}\,\frac{\beta}{\alpha}\,T^{\beta-\alpha}) \le 0.$$

Then $y=z$, this implies that the solution of the functional equation (3) is unique. Consequently, the solution of the problem (1) is unique.

4. Continuous dependence

Definition 1. [7] The solution x of the problem (1) depends continuously on the initial data x_0 and on the function f, if $\forall\epsilon>0,\ \exists\delta>0$ s.t.

$$\max\{\,|x_0-{x_0}^*|,|f-f^*|\}<\delta,$$

implies

$$\|x-x^*\|<\epsilon,$$

where x^* is the unique solution of the problem

$$\frac{dx^*(t)}{dt^\beta}=f^*(t,\frac{dx^*(t)}{dt^\alpha}),\quad a.e.\quad t\in(0,T],\quad x(0)={x_0}^*,\quad (5)$$

where $\beta,\alpha\in(0,1)$, and $\beta>\alpha$.

Theorem 3. Let the assumptions of Theorem 2 be satisfied, then the solution of the problem (1) depends continuously on the initial value x_0 and on the function f.

Proof. Let x, x^* be the two solutions of (1) and (5) respectively, then

$$|y(t)-y^*(t)| =$$

$$|\,\beta\,t^{\beta-1}f(t,\frac{1}{\alpha}\,t^{1-\alpha}\,y(t))-\beta\,t^{\beta-1}f^*(t,\frac{1}{\alpha}\,t^{1-\alpha}\,y^*(t))\,|$$

$$=\beta\,t^{\beta-1}\,|\,f(t,\frac{1}{\alpha}\,t^{1-\alpha}\,y(t))-f^*(t,\frac{1}{\alpha}\,t^{1-\alpha}\,y^*(t))\,|$$

$$=\beta\,t^{\beta-1}\,|\,f(t,\frac{1}{\alpha}\,t^{1-\alpha}\,y(t))-f(t,\frac{1}{\alpha}\,t^{1-\alpha}\,y^*(t))$$

$$+\,f(t,\frac{1}{\alpha}\,t^{1-\alpha}\,y^*(t))-f^*(t,\frac{1}{\alpha}\,t^{1-\alpha}\,y^*(t))\,|$$

$$\le\beta\,t^{\beta-1}\,|\,f(t,\frac{1}{\alpha}\,t^{1-\alpha}\,y(t))-f(t,\frac{1}{\alpha}\,t^{1-\alpha}\,y^*(t))\,|$$

$$+\,\beta\,t^{\beta-1}\,|\,f(t,\frac{1}{\alpha}\,t^{1-\alpha}\,y^*(t))-f^*(t,\frac{1}{\alpha}\,t^{1-\alpha}\,y^*(t))\,|$$

$$\le\beta\,t^{\beta-1}\,b\,|\,\frac{1}{\alpha}\,t^{1-\alpha}\,y(t)-\frac{1}{\alpha}\,t^{1-\alpha}\,y^*(t)\,|+\beta\,t^{\beta-1}\,\delta$$

$$\le\beta\,t^{\beta-1}\,b\,\frac{1}{\alpha}\,t^{1-\alpha}\,|y(t)-y^*(t)|+\beta\,t^{\beta-1}\,\delta$$

$$\le b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha}\,|y(t)-y^*(t)|+\beta\,t^{\beta-1}\,\delta$$

$$(1-b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha})\,|\,y(t)-y^*(t)|\le\beta\,t^{\beta-1}\,\delta$$

$$|\,y(t)-y^*(t)|\le\frac{\beta\,t^{\beta-1}\,\delta}{1-b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha}}.$$

Hence

$$\int_0^T|y(s)-y^*(s)|\,ds\le\frac{T^\beta\,\delta}{1-b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha}}.$$

Now,

$$|x(t)-x^*(t)|=|\,x_0+\int_0^t y(s)\,ds-{x_0}^*-\int_0^t y^*(s)\,ds\,|$$

$$=|\,(x_0-{x_0}^*)+\int_0^t(\,y(s)-y^*(s)\,)ds\,|$$

$$\le|x_0-{x_0}^*|+\int_0^t|y(s)-y^*(s)\,|ds$$

$$\le\delta+\frac{T^\beta\,\delta}{1-b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha}}$$

$$\|x-x^*\|\le\delta+\frac{T^\beta\,\delta}{1-b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha}}=\epsilon.$$

Which completes the proof.

5. Hyers-Ulam stability

Definition 2. Let the solution of the problem (1) be exists uniquely, then the problem (1) is Hyers-Ulam stable [5] if $\forall\epsilon>0,\ \exists\delta>0$ such that for any δ–approximate solution x_s of problem (1) satisfying

$$|\,\frac{dx_s(t)}{dt^\beta}-f(t,\frac{dx_s(t)}{dt^\alpha})\,|<\delta,$$

implies

$$\|x-x_s\|<\epsilon.$$

Theorem 4. Let the assumption of Theorem 2 be satisfied. Then the problem (1) is Hyers-Ulam stable.

Proof. Let

$$|\,\frac{dx_s(t)}{dt^\beta}-f(t,\frac{dx_s(t)}{dt^\alpha})\,|<\delta$$

$$-\delta<\frac{dx_s(t)}{dt^\beta}-f(t,\frac{dx_s(t)}{dt^\alpha})<\delta$$

$$-\delta<\frac{1}{\beta\,t^{\beta-1}}\frac{dx_s(t)}{dt}-f(t,\frac{1}{\alpha}t^{1-\alpha}\frac{dx_s(t)}{dt})<\delta$$

$$-\delta\,\beta\,t^{\beta-1}<\frac{dx_s(t)}{dt}-\beta\,t^{\beta-1}f(t,\frac{1}{\alpha}t^{1-\alpha}\frac{dx_s(t)}{dt})<\delta\,\beta\,t^{\beta-1}$$

Let $y_s(t)=\frac{dx_s(t)}{dt}$,

$$-\delta\,\beta\,t^{\beta-1}<y_s(t)-\beta\,t^{\beta-1}f(t,\frac{1}{\alpha}t^{1-\alpha}\,y_s(t))<\delta\,\beta\,t^{\beta-1}$$

$$|\,y_s(t)-\beta\,t^{\beta-1}f(t,\frac{1}{\alpha}t^{1-\alpha}\,y_s(t))\,|<\delta\,\beta\,t^{\beta-1}$$

Then

$$|y(t)-y_s(t)|=|\,\beta\,t^{\beta-1}f(t,\frac{1}{\alpha}t^{1-\alpha}y(t))-y_s(t)\,|$$

$$=|\,\beta\,t^{\beta-1}f(t,\frac{1}{\alpha}t^{1-\alpha}y(t))-(\,y_s(t)-\beta\,t^{\beta-1}f(t,\frac{1}{\alpha}t^{1-\alpha}\,y_s(t))\,)$$

$$-\,\beta\,t^{\beta-1}f(t,\frac{1}{\alpha}t^{1-\alpha}\,y_s(t))\,|$$

$$\le\beta\,t^{\beta-1}\,|\,f(t,\frac{1}{\alpha}t^{1-\alpha}y(t))-f(t,\frac{1}{\alpha}t^{1-\alpha}\,y_s(t))\,|$$

$$+\,|\,y_s(t)-\beta\,t^{\beta-1}f(t,\frac{1}{\alpha}t^{1-\alpha}\,y_s(t))\,|$$

$$\le\beta\,t^{\beta-1}\,b\,|\,\frac{1}{\alpha}t^{1-\alpha}y(t)-\frac{1}{\alpha}t^{1-\alpha}\,y_s(t)\,|+\delta\,\beta\,t^{\beta-1}$$

$$\le b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha}\,|y(t)-y_s(t)|+\delta\,\beta\,t^{\beta-1}$$

$$\le\frac{\delta\,\beta\,t^{\beta-1}}{1-b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha}}.$$

Thus

$$\int_0^T|y(\theta)-y_s(\theta)|\,d\theta\le\frac{\delta\,T^\beta}{1-b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha}}.$$

Now,

$$|x(t)-x_s(t)|=|\,x_0+\int_0^t y(\theta)\,d\theta-x_0-\int_0^t y_s(\theta)\,d\theta\,|$$

$$=|\,\int_0^t(\,y(\theta)-y_s(\theta)\,)\,d\theta\,|$$

$$\le\int_0^T|\,y(\theta)-y_s(\theta)\,|\,d\theta$$

$$\le\frac{\delta\,T^\beta}{1-b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha}}$$

$$\|x-x_s\|\le\frac{\delta\,T^\beta}{1-b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha}}=\epsilon.$$

Which completes the proof.

6. Examples

Example 1. Consider the following initial-value problem of the fractal differential equation

$$\frac{dx(t)}{dt^{\beta}} = 2t + \frac{1}{5}\frac{\frac{dx(t)}{dt^{\alpha}}}{1+\left|\frac{dx(t)}{dt^{\alpha}}\right|}, \quad a.e. \quad t \in (0,1], \quad x(0) = \frac{1}{2}. \quad (6)$$

Then

$$f(t,x) = 2t + \frac{1}{5}\frac{x(t)}{1+|x(t)|},$$

where $a(t) = 2t \in L1(I)$, $\beta = \frac{1}{2}$, $\alpha = \frac{1}{3}$ and $b = \frac{1}{5} > 0$ such that

$$\int_0^1 s^{\beta-1}|a(s)|ds = \frac{4}{3} \text{ and } b\,\frac{\beta}{\alpha}T^{\beta-\alpha} = \frac{3}{10} < 1.$$

We can show that all conditions of Theorem 2 are satisfied. Then, the initial-value problem (6) has a unique solution.

Example 2. Consider the following initial-value problem of the fractal differential equation

$$\frac{dx(t)}{dt^{\beta}} = \frac{1}{2}t^{1-\beta} + \frac{1}{3}\frac{\frac{dx(t)}{dt^{\alpha}}}{1+\sin^2 t}, \quad a.e. \quad t \in (0,2], \quad x(0) = \frac{1}{4}. \quad (7)$$

Then

$$f(t,x) = \frac{1}{2}t^{1-\beta} + \frac{1}{3}\frac{x(t)}{1+\sin^2 t},$$

where $a(t) = \frac{1}{2}t^{1-\beta} \in L1(I)$, $\beta = \frac{1}{4}$, $\alpha = \frac{1}{5}$ and $b = \frac{1}{3} > 0$ such that

$$\int_0^2 s^{\beta-1}|a(s)|ds = 1 \text{ and } b\,\frac{\beta}{\alpha}T^{\beta-\alpha} = 0.43 < 1.$$

We can show that all conditions of Theorem 2 are satisfied. Then, the initial-value problem (7) has a unique solution.

7. Conclusions

This research paper focuses on investigate the existence of solutions for the fractal differential problem (1) and properties associated with these solutions. Firstly, we examined the equivalence between the problem (1) and functional equation (3), then we studied the existence of at least one solution $x \in AC(I)$ of (1) by applying Schauder's fixed point theorem. Furthermore, we established sufficient conditions to ensure the uniqueness of the solution and its continuous dependence on the initial data x_0 and on the function f. We studied the Hyers-Ulam stability of the problem (1). Finally, some examples have been introduced to illustrate our results.

References

[1] Agarwal, R.P.; O'Regan, D. An Introduction to Ordinary Differential Equations. Springer New York; 2008.

[2] Barnsley, M.F.; Devaney, R.L.; Mandelbrot, B.B.; Peitgen, H.O.; Saupe, D.; Voss, R.F. The Science of Fractal Images. Springer New York; 1988.

[3] Curtain, R.F.; Pritchard, A.J. Functional Analysis in Modern Applied Mathematics; Academic Press: Cambridge, MA, USA, 1977.

[4] Deppman, A.; Meg´ıas, E.; Pasechnik, R. Fractal Derivatives, Fractional Derivatives and q-Deformed Calculus. Entropy. 2023, 25(7) 1008. https://doi.org/10.3390/e25071008.

[5] El-Sayed, A.M.A.; Ba-Ali, M.M.S.; Hamdallah, E.M.A. An Investigation of a Nonlinear Delay Functional Equation with a Quadratic Functional Integral Constraint. Mathematics. 2023, 11(21), 4475. https://doi.org/10.3390/math11214475.

[6] El-Sayed, A.M.A.; El-Sayed, W.G.; Nasim, S.I. On the Solvability of a Delay Tempered-Fractal Differential Equation. Journal of Fractional Calculus and Applications. 2024, 15(1) 1-14.

[7] Elsgolts, L.E. Differential Equations and the Calculus of Variations. 1970. Translated from the Russian by George Yankovsky. Mir Publishers.

[8] Goebel, K.; Kirk, W.A. Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge; 1990.

[9] Granas, A.; Dugundji, J. Fixed Point Theory. Springer Monographs in Mathematics. Springer New York; 2003.

[10] Hashem, H.H.G.; El-Sayed, A.M.A.; Al-Issa, S.M. Investigating Asymptotic Stability for Hybrid Cubic Integral Inclusion with Fractal Feedback Control on the Real Half-Axis. Fractal and Fractional. 2023, 7(6), 449; https://doi.org/10.3390/fractalfract7060449.

[11] Mandelbrot, B.B. Fractal Geometry: what is it, and what does it do? Proceedings of the Royal Society of London A Mathematical and Physical Sciences. 1989, 423(1864) 3–16.

[12] Parvate, A.; Gangal, A.D. Fractal Differential Equations and Fractal-Time Dynamical Systems. Pramana. 2005, 64(3) 389–409.

[13] Peitgen, H.O.; Jurgens, H.; Saupe, D. Chaos and Fractals. Springer New York; 2004.

[14] Steeb, W.H. Problems and Solutions. WORLD SCIENTIFIC, 2016.