A constrained problem of a nonlinear functional integral equation subject to the pantograph problem

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KEYWORDS: Pantograph differential equation, Hyers-Ulam stability, constrained problem, existence of solution.

Received: December 21, 2023
Accepted: January 12, 2024
Published: January 22, 2024

ABSTRACT: Here we study the existence of solution and its continuous dependence of a constrained problem of a nonlinear functional integral equation subject to a constraint of an initial value problem of a pantograph differential equation. The Hyers-Ulam stability of the problem will be proved.

1. INTRODUCTION

Differential and integral equations are crucial in nonlinear analysis. Many fundamental laws of physics and chemistry can be formulated as differential and integral equations. In biology and economics, differential equations are used to model the behavior of complex systems. Many authors are concerned with the study of this kind of equations see [2-5-6-9-10-16]. Pantograph equation is a delay differential equation (DDE) arising in electrodynamics. This type of equations have numerous applications in most fields, see [15-17-18-21-22-23]. Constrained problems are essential in the mathematical depiction of real-world situations, where such problems are transformed into mathematical models. The relevance of handling constraints or control variables arises from the unanticipated elements that persistently disrupt biological systems in the real world; biological traits like survival rates might change as a result. The question of whether an ecosystem can survive those erratic, disruptive occurrences that happen for a short while is of practical significance to ecology, see [1-3-4-7-8-11-12-13-14-15-19-20].

Now let $\gamma$, $\beta$ $\in$ $(0, 1)$, $\lambda >0$. Let $C[0,T]$ be the class of continuous function defined on $[0,T]$, the norm of $x \in C[0,T]$ is given by.

$$
\|x\| = \sup_{t \in [0,T]} |x(t)|
$$

Consider the nonlinear functional integral equation

$$
y(t) = f_1 \left( t, \lambda \int_0^t g(s,y(s),u(s)) \, ds \right), \ t \in [0,T]. \tag{1}
$$

$$
du/dt = f_2 \left( t, u(t), u(\xi t) \right), \ a.e. \ t \in (0,T] \text{ and } u(0) = u_0. \tag{2}
$$

Here, Firstly, we prove the existence of a unique solution $u \in C [0,T]$ of the problem (2) and study the continuous dependence of the solution $u$ on $\gamma$ and $u_0$. Secondly, we prove the existence of a unique solution of the integral equation (1) and study the continuous dependence of $y$ on $\beta$, $\lambda$. Finally, we study Hyers-Ulam stability of our problem (1), (2).

2. Existence of solution

Consider the following assumptions

1) $f_1 : [0,T] \times R \rightarrow R$ is continuous and satisfies the Lipchitz condition.

$$
|f_1 (t, x) - f_1 (t, \bar{x})| \leq k_1 |x - \bar{x}|
$$

2) $f_2 : [0,T] \times R \times R \rightarrow R$ is measurable in $t \in [0,T]$ for all $u \in R$ and satisfies the Lipchitz condition.

$$
|f_2 (t, u, u_2) - f_2 (t, \bar{u}, \bar{u_2})| \leq k_2 (|u_1 - \bar{u}_1| + |u_2 - \bar{u}_2|)
$$

3) $g : [0,T] \times R \times R \rightarrow R$ is measurable in $t \in [0,T]$ for all $y$ and $u \in R$ and satisfies the lipschitz condition.

$$
|g(t, y, u) - g(t, \bar{y}, \bar{u})| \leq k_3 (|y - \bar{y}| + |u - \bar{u}|)
$$

4) Let $k = \max \{k_1,k_2,k_3\}$. 

10.21608/AJST.2024.257068.1023
Remark

From (3) we have

(i) $|f_1(t,x)| - |f_1(t,0)| \leq |f_1(t,x) - f_1(t,0)| \leq k |x|

and $|f_1(t,x(t))| \leq k |x(t)| + f^*_1$, $f^*_1 = \sup_{t\in[0,T]} |f_1(t,0)|$.

Also, from (4) and (5) we can get

(ii) $|f_2(t,u,\bar{u})| \leq k (|u(t)| + |\bar{u}(t)|) + f^*_2$,

$f^*_2 = \sup_{t\in[0,T]} |f_2(t,0,0)|$.

(iii) $|g(t,y,u)| \leq k (|y(t)| + |u(t)|) + g^*$,

$g^* = \sup_{t\in[0,T]} |g(t,0,0)|$.

Now, we study the problem (2).

2.1 The problem (2)

Here we study the initial value problem (2)

Theorem 1

Let the assumption (2) be satisfied, if $2kT < 1$, then there exists a unique solution $u \in C[0,T]$ of the problem (2).

Proof.

Integrating (2), we obtain

$$u(t) = u_0 + \int_0^t f_2(s,u(s),u(\psi s))ds, \quad t \in [0,T].$$

Differentiating (6) we obtain (2) and from (2) we deduce $u(0) = u_0$.

Define the operator $F$ by

$$F(u)(t) = u_0 + \int_0^t f_2(s,u(s),u(\psi s))ds.$$ (7)

Let $u, \bar{u} \in C[0,T]$ be two solutions of equation (6), then

$$|F(u(t)) - F(\bar{u}(t))| \leq |\int_0^t f_2(s,u(s),u(\psi s))ds - \int_0^t f_2(s,\bar{u}(s),\bar{u}(\psi s))ds|$$

$$\leq k |\int_0^t [u(s) - \bar{u}(s)]ds + \int_0^t [u(\psi s) - \bar{u}(\psi s)]ds|$$

$$\leq k T ||u - \bar{u}|| + k T ||u(\psi s) - \bar{u}(\psi s)||$$

$$\leq 2k T ||u - \bar{u}||.$$ (8)

Then we obtain

$$|F(u(t)) - F(\bar{u}(t))| \leq \epsilon.$$ (9)

Theorem 2

Let the assumptions of Theorem 1 be satisfied, then the solution $u \in C[0,T]$ of (2) depends continuously on $u_0$ and $\psi$.

Proof.

Where

$$u^*(t) = u_0^* + \int_0^t f_2(t,u^*(s),u^*(\psi s))ds.$$ (10)

Definition 2

Let the solution of (2) be exists, then approximate problem (2) is Hyers-Ulam stable if $\forall \epsilon > 0$, $\exists \delta(\epsilon) > 0$ and any solution $u_0$ of (2) satisfying

$$|u(t) - u^*(t)| = |\int_0^t f_2(s,u(s),u(\psi s))ds - u_0^* - \int_0^t f_2(s,u_0^*,u^*(\psi s))ds|$$

$$\leq \delta_1 + |\int_0^t f_2(s,u(s),u(\psi s))ds - \int_0^t f_2(s,u^*(s),u^*(\psi s))ds|$$

$$\leq \delta_1 + k T ||u - u^*|| + k T ||u(\psi s) - u^*(\psi s)||$$

$$\leq \delta_1 + k T ||u - u^*|| + k T ||u(\psi s) - u^*(\psi s)||.$$ (11)

Then

$$||u - u^*|| \leq \delta_1 + k T \epsilon.$$

Theorem 3

Let the assumptions of Theorem 1 be satisfied, then the problem (2) is Hyers-Ulam stable.

Proof.

Integrating both sides of (8), we obtain

$$-\delta T < u(t) - u_0 - \int_0^t f_2(\theta,u_0(\theta),u_0(\psi \theta))d\theta < \delta T$$

Now,

$$|u(t) - u_0(\psi t)| = |u_0 + \int_0^t f_2(\theta,u_0(\theta),u_0(\psi \theta))d\theta - u_0(\psi t)|$$

$$\leq |u_0 + \int_0^t f_2(\theta,u_0(\theta),u_0(\psi \theta))d\theta - \int_0^t f_2(\theta,u_0(\psi \theta))d\theta - u_0(\psi t)|$$

$$\leq \delta T + k T ||u(\psi t) - u^*(\psi t)||.$$ (12)

Then

$$(1 - 2kT)||u - u_0|| \leq \delta T.$$
and
\[ |u - u_\delta| \leq \frac{\delta T}{1 - 2kT} = \varepsilon. \]

2.2 The initial value problem (1)

Theorem 4
Let the assumptions 1, 2 and 4 be satisfied, let \( u \) be the solution of (2), if \( \lambda \beta T < 1 \), then the problem (1) has a unique solution \( x \in C[0,T] \).

**Proof.**
Define the operator \( F \) by
\[ F_y(t) = f(t, \lambda \int_0^t g(s, y(s), u(s)) ds \) (9)\]
Let \( y \in C[0,T] \), and for \( t_2, t_1 \in [0,T] \) such that \( t_2 - t_1 < \delta \), then we have
\[ |F_1(t_2, y(t_2)) - F_1(t_1, y(t_1))| = |f_1(t_2, \lambda \int_0^{t_2} g(s, y(s), u(s)) ds) - f_1(t_1, \lambda \int_0^{t_1} g(s, y(s), u(s)) ds)| \]
\[ \leq |f_1(t_2, \lambda \int_0^{t_2} g(s, y(s), u(s)) ds) - f_1(t_1, \lambda \int_0^{t_1} g(s, y(s), u(s)) ds)| + f_1(t_2, \lambda \int_0^{t_2} g(s, y(s), u(s)) ds) - f_1(t_1, \lambda \int_0^{t_1} g(s, y(s), u(s)) ds)| \]
\[ \leq \lambda \int_0^{t_1} |g(s, y(s), u(s))| ds + \delta \]
\[ \leq \delta + \lambda k^2 \int_{t_1}^{t_2} |g(s, y(s), u(s))| ds + \lambda \int_{t_1}^{t_2} g(s, y(s), u(s))| ds \]
Now, we prove that \( F \) is contraction. Let \( y, y' \) be two solutions of (1), then
\[ |F_1(t_2, y(t_2)) - F_1(t_1, y(t_1))| = \left| f_1(t_2, \lambda \int_0^{t_2} g(s, y(s), u(s)) ds) - f_1(t_1, \lambda \int_0^{t_1} g(s, y(s), u(s)) ds) \right| \]
\[ \leq \lambda k \int_0^{t_1} |g(s, y(s), u(s)) - g(s, y'(s), u(s))| ds \]
\[ \leq \lambda k^2 \int_0^{t_1} |y(s) - y'(s)| ds \]
\[ \leq \lambda k^2 T \|y - y'\|. \]
Then \( F \) is contraction [5] and (1) has a unique solution \( y \in C[0,T] \).

2.3 Continuous dependence

**Definition 3**
The solution \( y \in C[0,T] \) of (1) depends continuously on \( \lambda, \beta \) and \( u \) if \( \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \) such that
\[ \max \{|u - u_\delta|, |\beta - \beta'\|, |\beta - \beta'\| \} < \delta. \]
Then
\[ \|y - y'\| < \varepsilon, \]
Where \( y' \) is the solution of (1)
\[ y'(t) = f_1(t, \lambda \int_0^{t_2} g(s, y'(s), u'(s)) ds). \]

**Theorem 5** Let the assumptions of Theorem 1,2,3 be satisfied, then the solution \( y \in C[0,T] \) of (1) depends continuously on \( \lambda, \beta, u' \).

**Proof.**
\[ |y(t) - y'(t)| = |f_1(t, \lambda \int_0^{t_2} g(s, y(s), u(s)) ds - f_1(t, \lambda \int_0^{t_2} g(s, y'(s), u'(s)) ds)| \]
\[ \leq k \lambda \int_0^{t_2} |g(s, y(s), u(s)) - g(s, y'(s), u'(s))| ds \]
\[ \leq k \lambda \int_0^{t_2} |y(s) - y'(s)| ds \]
\[ \leq k \lambda \int_0^{t_2} |g(s, y(s), u(s)) - g(s, y'(s), u'(s))| ds \]
\[ \leq k \lambda \int_0^{t_2} |y(s) - y'(s)| ds \]
Now,
\[ \leq k \lambda |y - y'| + k^2 \Delta \|\beta\| \|y - y'\| \]
Then
\[ (1 - k^2 \Delta \|\beta\|) \|y - y'\| \leq k \lambda |y - y'| + k^2 \Delta \|\beta\| \|y - y'\| \]
Then
\[ \|y - y'\| \leq \frac{k \lambda \Delta \|\beta\|}{1 - k^2 \Delta \|\beta\|}. \]

**Definition 4**
Let the solution of (1) be exists then the problem (1) is Hyers-Ulam stable if \( \forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \) and any approximate solution \( y_\delta \) of (1) satisfying
\[ |y(t) - f_1(t, \lambda \int_0^{t_2} g(s, y_\delta(s), u_\delta(s)) ds)| < \varepsilon. \]
Then
\[ \|y - y\| < \varepsilon, \]
Where
\[ -\delta < y_\delta(t) - f_1(t, \lambda \int_0^{t_2} g(s, y_\delta(s), u_\delta(s)) ds) < \delta. \]

**Theorem 6** Let the assumptions of Theorem 4 be satisfied, then the problem (1) is Hyers-Ulam stable.

**Proof.**
\[ |y(t) - y(t)| = |f_1(t, \lambda \int_0^{t_2} g(s, y(s), u(s)) ds - y(t)| \]
\[ = |f_1(t, \lambda \int_0^{t_2} g(s, y(s), u(s)) ds - f_1(t, \lambda \int_0^{t_2} g(s, y_\delta(s), u_\delta(s)) ds)| \]
\[ + f_1(t, \lambda \int_0^{t_2} g(s, y_\delta(s), u_\delta(s)) ds - y(t)) - y(t)| \]
\[ \leq + f_1(t, \lambda \int_0^{t_2} g(s, y_\delta(s), u_\delta(s)) ds - y(t)) - y(t)| \]
\[ \leq \delta + f_1(t, \lambda \int_0^{t_2} g(s, y(s), u(s)) ds - y(t)) - y(t)| \]
\[ \leq \delta + k \lambda \int_0^{t_2} |g(s, y(s), u(s)) - g(s, y_\delta(s), u_\delta(s))| ds \]
\[ \leq \delta + k \lambda \int_0^{t_2} |y(s) - y_\delta(s)| ds \]
Then
\[ \|y - y\| \leq \delta + k \lambda \|\beta\| \|y - y\|. \]
\[ (1-\lambda, k^2 \beta T) \| y - y_0 \| \leq \delta \]

\[ \| y - y_0 \| \leq \frac{\delta}{1-\lambda k^2 \beta T} = \varepsilon. \]

**Example**

Consider the following example

\[ y(t) = \frac{1}{5} e^{-t} \cos^2 t + \frac{1}{4} x, \quad \text{thus} \quad |y_1(t) - y_2(t)| \leq \frac{1}{4} |x - \bar{x}| \]

\[ g(s, y, u) = \frac{e^{-s} + \frac{1}{3} y(s) + \frac{1}{5} u(s)}{\gamma(s) - g(s, \bar{y}, \bar{u})} \leq \frac{1}{3} \gamma(s) - g(s, \bar{y}, \bar{u}) \]

\[ f_2(t, u(t), u(\tau)) = \frac{\ln(1+t)}{2} + \frac{e^{-t}}{5} u(t) + \frac{1}{5} u(t), \quad \text{thus} \]

\[ |f_2(t, u_1(t) - u_1(t), u_2(t) - u_2(t))| \leq \frac{1}{5} |u_1(t) - u_1(t)| + 2kT = \frac{2}{3} < 1 \]

Clear all assumptions of Theorem 1 is satisfied, thus problem (10)-(11) has unique solution.

References:


