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Article

On some perturbed models

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ABSTRACT: This paper explores novel concepts within perturbed models, focusing on their local stability analysis of fixed points. The investigation involves numerical simulations employing bifurcation diagrams and phase diagrams to substantiate findings and delineate intricate dynamics. By leveraging these computational tools, the research aims to validate its results comprehensively. Moreover, the theoretical implications of these new concepts are thoroughly scrutinized and compared with existing frameworks. This comparative analysis sheds light on the advancements introduced by the proposed models, highlighting their potential contributions to the field. Through rigorous examination and validation via simulations and theoretical scrutiny, the study not only confirms the stability properties of fixed points under perturbations but also elucidates the broader implications of these findings. Furthermore, the utilization of bifurcation and phase diagrams serves to illustrate the complex behaviors and transitions observed within the models, offering a visual representation that enhances the understanding of the dynamics involved. Overall, this paper contributes to advancing the understanding of perturbed models by integrating theoretical insights with numerical validation, thus paving the way for future research in this area.

1. INTRODCTION

Perturbation theory plays a crucial role in dynamical systems, facilitating the approximation of solutions to analytically intractable problems. It is extensively employed in the investigation of stability, resonance phenomena, and bifurcations across physical, biological, and engineering systems [1-3].

Dynamical systems theory studies the behavior of systems over time, offering a powerful framework for modeling complex phenomena in physics, biology, economics, and engineering. It enables the analysis of dynamic behavior through differential equations, bifurcations, chaos theory, and stability analysis. By exploring these concepts, researchers gain insights into the intricate dynamics of natural and artificial systems, unraveling the principles that govern complex phenomena and enhancing our understanding of the world [4-6].

Bifurcation, a fundamental concept in dynamical systems theory, refers to qualitative changes in system behavior as parameters vary. It can lead to complex behaviors like chaos, periodic orbits, or stable equilibria. Understanding bifurcations is essential for predicting and controlling system dynamics and exploring the boundaries between order and chaos [7-8].

Chaos theory, a captivating aspect of dynamical systems, investigates systems that exhibit seemingly random and unpredictable behavior. It uncovers underlying order and patterns, highlighting sensitivity to initial conditions. Chaos theory profoundly impacts the understanding of complex systems in nature and society, such as weather patterns, population dynamics, and financial markets [9-10].

2. The First Model:

Consider the following equation**:**

$$x_{n+1} = 1 - \rho x_{n-1}^2, \qquad n = 0,1,2,... \tag{2.1}$$

$$x(0) = x_0, \qquad x(-1) = x_{-1},$$

let $y_n = x_{n-1}$ then,

$$x_{n+1} = 1 - \rho y_n^2,$$

$$y_{n+1} = x_n.$$

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2.1. First Case of Perturbation

Let there exist a perturbation as

$$y_{n+1} = a x_n + \grave{o} y_n .$$

Then the model (2.1) can be written as

$$\begin{aligned} x_{n+1} &= 1 - \rho y_n^2 , \\ y_{n+1} &= a x_n + \grave{o} y_n , \\ x(0) &= x_0, \quad y(0) = y_0 . \end{aligned}$$

2.1.1. Fixed Points

$$\begin{aligned} x &= 1 - \rho y_n^2 , \\ y &= a x_n + \grave{o} y_n . \end{aligned}$$

Therefore, we have two fixed points (x_1^*, y_1^*) and (x_2^*, y_2^*) where,

$$x_1^* = -\frac{(\frac{1-\grave{o}}{a})^2 + \sqrt{(\frac{1-\grave{o}}{a})^4 + 4\rho(\frac{1-\grave{o}}{a})}}{2\rho}, \quad y_1^* = -\frac{(\frac{1-\grave{o}}{a}) + \sqrt{(\frac{1-\grave{o}}{a})^2 + 4\rho(\frac{1-\grave{o}}{a})}}{2\rho}$$

$$x_2^* = -\frac{(\frac{1-\grave{o}}{a}) - \sqrt{(\frac{1-\grave{o}}{a})^2 + 4\rho(\frac{1-\grave{o}}{a})}}{2\rho}, \quad y_2^* = -\frac{(\frac{1-\grave{o}}{a}) - \sqrt{(\frac{1-\grave{o}}{a})^2 + 4\rho(\frac{1-\grave{o}}{a})}}{2\rho}$$

2.1.2. Stability Analysis

The Jacobin matrix is given by

$$J(x_1^*, y_1^*) = \begin{bmatrix} 0 & -2\rho y_1^* \\ a & \grave{o} \end{bmatrix}.$$

The characteristic equation is given by

$$\lambda^2 - \grave{o}\lambda + [(\grave{o}-1) + \sqrt{(1-\grave{o})^2 + 4\rho a^2}] = 0.$$

Lemma 2.1. [11]

Let $F(\lambda) = \lambda^2 + P\lambda + Q$ be the characteristic equation of eigenvalues associated to the Jacobin matrix evaluated at a fixed point (x^*, y^*) then (x^*, y^*) is

1. A sink if $\lambda_1 < 1$ and $\lambda_2 < 1$,
2. A source if $\lambda_1 > 1$ and $\lambda_2 > 1$,
3. A saddle if $\lambda_1 > 1$ and $\lambda_2 < 1$ or ($\lambda_1 < 1$ and $\lambda_2 > 1$),
4. A non-hyperbolic if either $\lambda_1 = 1$ *or* $\lambda_2 = 1$.

Lemma 2.2. [11]

Let $F(\lambda) = \lambda^2 + P\lambda + Q$ suppose that $F(1) > 0$ and $F(\lambda) = 0$ has two roots λ_1 and λ_2 then

1. $F(-1) > 0$ and $Q < 1$ if and only if $\lambda_1 < 1$ and $\lambda_2 < 1$,
2. $F(-1) < 0$ if and only if $\lambda_1 > 1$ and $\lambda_2 < 1$ or ($\lambda_1 < 1$ and $\lambda_2 > 1$),
3. $F(-1) > 0$ and $Q > 1$ if and only if $\lambda_1 > 1$ and $\lambda_2 > 1$.

By applying lemma (2.1) and lemma (2.2), we deduce that the model (2.1) is

1. Stable if $[(\grave{o}-1) + \sqrt{(1-\grave{o})^2 + 4\rho a^2}] < 1$,
2. Unstable if $[(\grave{o}-1) + \sqrt{(1-\grave{o})^2 + 4\rho a^2}] > 1$.

2.2. Second Case of Perturbation

Let $y_n = -x_{n-1}$ in equation (2.1). Moreover, let there exist a perturbation as

$$y_{n+1} = -a x_n + \grave{o} y_n .$$

Then, the model (2.1) can be written as

$$\begin{aligned} x_{n+1} &= 1 - \rho y_n^2 , \\ y_{n+1} &= -a x_n + \grave{o} y_n , \\ x(0) &= x_0, \quad y(0) = y_0 \end{aligned}$$

2.2.1. Fixed Points

$$\begin{aligned} x &= 1 - \rho y_n^2 , \\ y &= -a x_n + \grave{o} y_n . \end{aligned}$$

Therefore, we have two fixed points (x_1^*, y_1^*) and (x_2^*, y_2^*) where,

$$x_1^* = \frac{\grave{o}-1}{a}\left(\frac{(\frac{\grave{o}-1}{a}) + \sqrt{(\frac{\grave{o}-1}{a})^2 + 4\rho}}{2\rho}\right), \quad y_1^* = \frac{(\frac{\grave{o}-1}{a}) + \sqrt{(\frac{\grave{o}-1}{a})^2 + 4\rho}}{2\rho},$$

$$x_2^* = \frac{\grave{o}-1}{a}\left(\frac{(\frac{\grave{o}-1}{a}) - \sqrt{(\frac{\grave{o}-1}{a})^2 + 4\rho}}{2\rho}\right), \quad y_2^* = \frac{(\frac{\grave{o}-1}{a}) - \sqrt{(\frac{\grave{o}-1}{a})^2 + 4\rho}}{2\rho}.$$

2.2.2. Stability Analysis

The Jacobin matrix is given by

$$J(x_1^*, y_1^*) = \begin{bmatrix} 0 & -2\rho y_1^* \\ -a & \grave{o} \end{bmatrix}.$$

The characteristic equation is given by

$$\lambda^2 - \grave{o}\lambda + [(\grave{o}-1) + \sqrt{(\grave{o}-1)^2 + 4\rho a^2}] = 0.$$

By applying lemma (2.1) and lemma (2.2), we deduce that the model (2.1) is

Stable if $[(\grave{o}-1) - \sqrt{(1-\grave{o})^2 + 4\rho a^2}] < 1$, see (**Figures 1&2**).

3. The Second Model

Consider the following equation

$$x_{n+1} = 1 - \rho x_n x_{n-1}, \qquad n = 0,1,2,\ldots$$
$$x(0) = x_0, \qquad x(-1) = x_{-1}, \qquad (3.1)$$

let $y_n = x_{n-1}$ then,

$$x_{n+1} = 1 - \rho x_n y_n,$$
$$y_{n+1} = x_n.$$

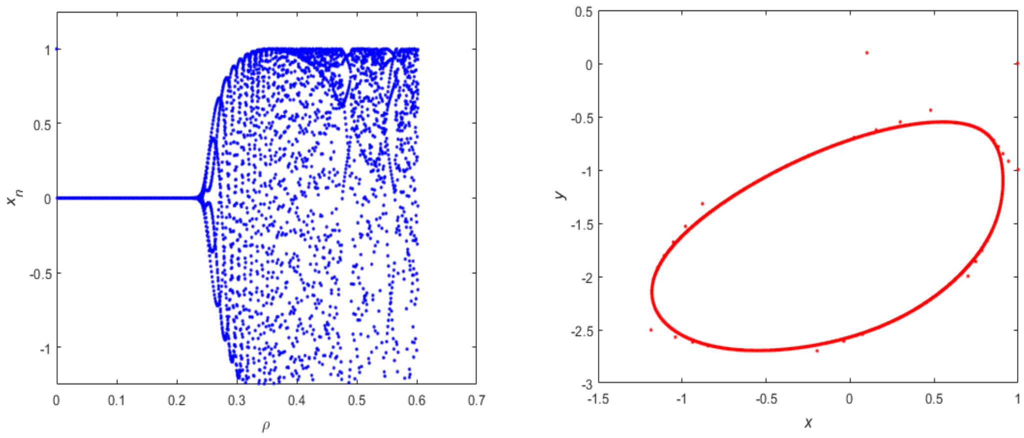

Figure 1. Bifurcation and phase diagram of model (2.1) at a=1, eps=1.

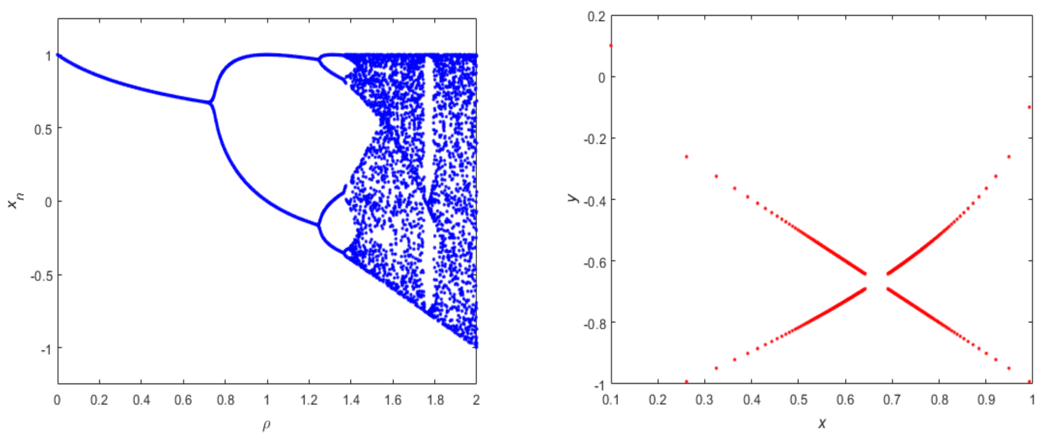

Figure 2. Bifurcation and phase diagram of model (2.1) at a=1, eps=0.

3.1. First Case of Perturbation

Let there exist a perturbation as

$$y_{n+1} = a x_n + \epsilon y_n + c.$$

Then the model (3.1) can be written as

$$x_{n+1} = 1 - \rho x_n y_n,$$
$$y_{n+1} = a x_n + \epsilon y_n + c,$$
$$x(0) = x_0, \quad y(0) = y_0$$

3.1.1. Fixed Points

$$x = 1 - \rho x_n y_n ,$$
$$y = ax_n + y_n + c$$

Therefore, we have two fixed points (x_1^*, y_1^*) and (x_2^*, y_2^*) where,

$$x_1^* = \frac{-(1-\grave{o}) + \sqrt{(1-\grave{o})^2 - 4\rho[\rho c - (1-\grave{o})]}}{2\rho}, y_1^* = \left(\frac{1}{1-\grave{o}}\right)\left[\frac{-(1-\grave{o}) + \sqrt{(1-\grave{o})^2 - 4\rho[\rho c - (1-\grave{o})]}}{2\rho} + c\right]$$

$$x_2^* = \frac{-(1-\grave{o}) - \sqrt{(1-\grave{o})^2 - 4\rho[\rho c - (1-\grave{o})]}}{2\rho}, y_2^* = \left(\frac{1}{1-\grave{o}}\right)\left[\frac{-(1-\grave{o}) - \sqrt{(1-\grave{o})^2 - 4\rho[\rho c - (1-\grave{o})]}}{2\rho} + c\right]$$

3.1.2 Stability Analysis

The Jacobin matrix is given by

$$J(x_1^*, y_1^*) = \begin{bmatrix} -\rho y_1^* & -\rho x_1^* \\ 1 & \grave{o} \end{bmatrix}.$$

The characteristic equation is given by

$$\lambda^2 + \left(\rho y_1^* - \grave{o}\right)\lambda + [\rho x_1^* - \grave{o}\rho y_1^*] = 0.$$

By applying lemma (2.1) and lemma (2.2), we deduce that the model (3.1) is

1. Stable if $[\rho x_1^* - \grave{o}\rho y_1^*] < 1$,
2. Unstable if $[\rho x_1^* - \grave{o}\rho y_1^*] > 1$.

3.2. Second Case of Perturbation

Let $y_n = -x_{n-1}$ in equation (3.1). Moreover, let there exist a perturbation as

$$y_{n+1} = -ax_n + \grave{o}y_n + c.$$

Then, the model (3.1) can be written as

$$x_{n+1} = 1 + \rho x_n y_n ,$$
$$y_{n+1} = -ax_n + \grave{o}y_n + c,$$
$$x(0) = x_0 , y(0) = y_0$$

3.2.1. Fixed Points

$$x = 1 + \rho x_n y_n$$
$$y = -ax_n + \grave{o}y_n + c$$

Therefore, we have two fixed points (x_1^*, y_1^*) and (x_2^*, y_2^*) where,

$$x_1^* = \frac{(1-\grave{o}) + \sqrt{(1-\grave{o})^2 - 4\rho[\rho c - (1-\grave{o})]}}{2\rho}, y_1^* = \left(\frac{1}{1-\grave{o}}\right)\left[-\left\langle\frac{(1-\grave{o}) + \sqrt{(1-\grave{o})^2 - 4\rho[\rho c - (1-\grave{o})]}}{2\rho}\right\rangle + c\right]$$

$$x_2^* = \frac{(1-\grave{o}) - \sqrt{(1-\grave{o})^2 - 4\rho[\rho c - (1-\grave{o})]}}{2\rho}, y_2^* = \left(\frac{1}{1-\grave{o}}\right)\left[-\left\langle\frac{(1-\grave{o}) - \sqrt{(1-\grave{o})^2 - 4\rho[\rho c - (1-\grave{o})]}}{2\rho}\right\rangle + c\right]$$

3.2.2. Stability Analysis

The Jacobin matrix is given by

$$J(x_1^*, y_1^*) = \begin{bmatrix} \rho y_1^* & \rho x_1^* \\ 1 & \grave{o} \end{bmatrix}.$$

The characteristic equation is given by

$$\lambda^2 - \left(\rho y_1^* - \grave{o}\right)\lambda + [\rho x_1^* - \grave{o}\rho y_1^*] = 0$$

By applying lemma (2.1) and lemma (2.2), we deduce that the model (3.1) is

1. Stable if $[\rho x_1^* + \grave{o}\rho y_1^*] < 1$, see (**Figures 3&4**).
2. Unstable if $[\rho x_1^* + \grave{o}\rho y_1^*] > 1$, see (**Figures 3&4**).

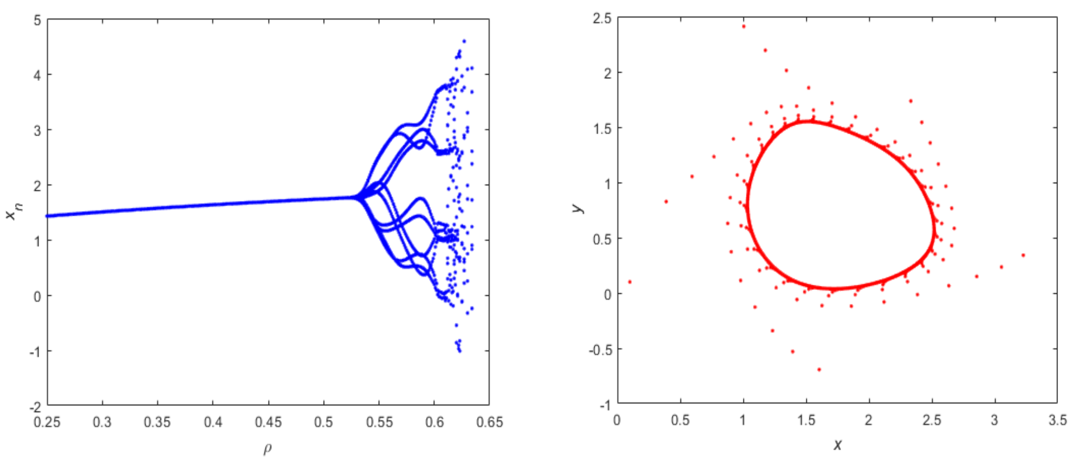

Figure 3. Bifurcation and phase diagram of model (3.1) at a=1, eps=0.1.

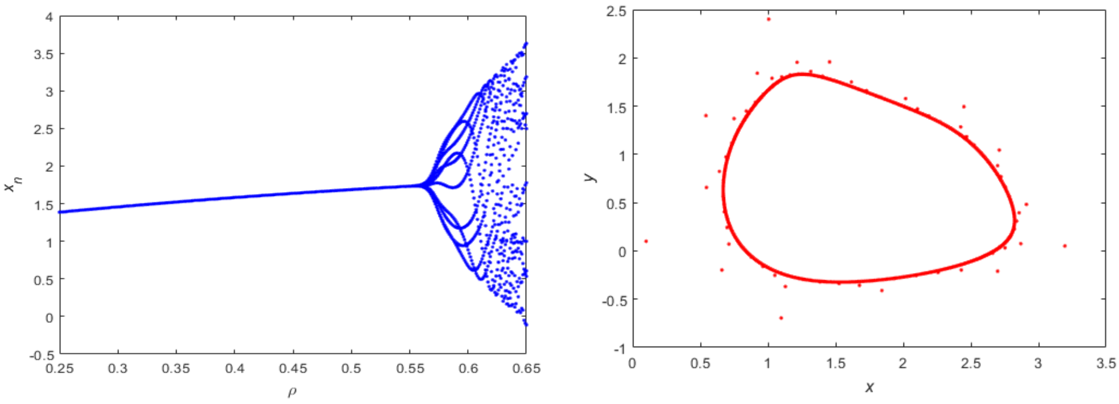

Figure 4. Bifurcation and phase diagram of model (3.1) at a=1, eps=0.

Conclusion

In conclusion, this paper discussed new concepts of some perturbed models. The local stability analysis of the fixed points was provided. The utilization of numerical simulations, including bifurcation and phase diagrams, substantiates the theoretical findings and reveals additional complex dynamical behaviors. The comparative analysis of the theoretical outcomes underscores the significance and applicability of these new concepts, contributing to a deeper understanding of the intricate dynamics within perturbed systems.

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