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Article

The influence of a fractional order parameter on the deformation of the 1D thermoviscoelastic cylinder cavity problem

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ABSTRACT: This study aims to investigate the effects of a non-singular fractional derivative on the thermomechanical responses of a one-dimensional thermoviscoelastic cylinder cavity problem. Using the Mittag-Leffler function as a relaxation function in mathematical models provides a more precise and comprehensive depiction of the behavior of thermo-viscoelastic materials. The boundary surfaces of the cylinder cavity are traction-free and associated with thermal shock. The analytical solution in the transformed domain is derived using a combination of direct approaches and Laplace transform techniques. We present a systematic study of traveling discontinuities in a hereditary thermoelastic space at different values of fractional order parameter α. The study shows that waves in this model travel at finite speeds. This aspect is new for fractional models. Finally, we have constructed a conclusion for a specific problem based on the numerical results and accompanying 3D graphics, as well as discussed, the effect of the deformation on the area of the cross-section of the cylinder cavity.

1. INTRODCTION

The growing use and progress of polymers and composite materials in recent years has guaranteed the ongoing relevance of researching linear viscoelastic theory. Materials with both viscous and elastic properties are studied in terms of viscoelasticity. After the force producing distortion is removed, an elastic material returns to its former shape; a viscous substance does the opposite. These materials must show that the applied force and the deformation that results are clearly correlated in order to be quantified. Time, temperature, and loading rate affect the rheological properties of linear viscoelastic materials. The convolutional integral, which considers input history including both current and past inputs, may be used to define the viscoelastic response.

Time-dependent material characteristics have been thoroughly studied by Tschoegl [1]. The linear viscoelastic behavior as described by a mechanical model was provided by Gross [2]. By their study, Atkinson and Craster [3] have improved our knowledge of fracture mechanics and its application in viscoelastic materials. Non-linear theory is well studied in the work of Rajagopal and Saccomandi [4].

An important basis for modeling many physical and engineering processes has been the basic idea of the Fourier law of heat conduction [5]. Known as "second sound," some solid conductors transmit heat over time with wave-like behavior as opposed to the more typical diffusion. This phenomenon has been carefully studied and recorded by the literature [5] and its associated sources. Consequently, wave-like events cannot be explained by the diffusivity that is present in the Fourier equation across all time scales. As such, it is uncertain whether the short-time domain heat transfer studies can be well simulated by the Fourier law. Such situations need either a different constitutive rule or a quantitative adjustment. Known by most as the telegrapher equation, the Maxwell-Cattaneo-Vernotte equation is a mathematical formula that seamlessly transitions from wave-like to diffusion-like behavior by including a time component. Long term, this equation approaches the traditional Fourier law and short term, it faithfully simulates a wave with a restricted speed. Generalized thermoelasticity with a single relaxation time was initially proposed by Lord and Shulman [6] to characterize the behavior

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of an isotropic body. Sherief et al. [7-10] investigate the theoretical study of thermal and elastic reactions in an infinite-length cylinder under thermal and mechanical stresses. They use the generalized thermoelasticity framework to improve prediction accuracy by including thermal relaxation time.

The coupled theory of thermo-viscoelasticity was introduced by Kovalenko and Karnaukhov in [11]. El-Karamany and Ezzat [12] and Ezzat et al. [13] have developed generalized thermo-viscoelasticity models with one or two relaxation times, but without consideration for volume relaxation effects. Ezzat and El Karamany [14] have made theoretical contributions to the field, including uniqueness theorem proofs in various situations. Furthermore, El-Karamany and Ezzat [15] have investigated the extended thermo-viscoelastic theory's propagation of solution discontinuities. El-Karamany and Ezzat [16, 17] used a linear micropolar electro-magnetic thermoelasticity model with two relaxation times to come up with the idea of boundary elements and the reciprocity and uniqueness theorems. Ezzat et al. [18] investigated the effects of relaxation on the volume characteristics of an electrically conducting viscoelastic material. Ezzat and Awad [19] develop a linear theory of micropolar thermo-viscoelasticity with mass diffusion, proving related theorems.

Many current models of physical events have been proven to be changed using fractional calculus. It has been shown by Caputo & Mainardi [20, 21] and Caputo [22] that the description of viscoelastic materials using fractional derivatives matches experimental data well. Moreover, they have shown a relationship between linear viscoelasticity principles and fractional derivatives. Many studies have produced fractional heat conduction models [23-33] and fractional viscoelastic models [34, 35]. The behavior of thermo-viscoelastic materials is described mathematically by Ezzat et al. [36], Hendy [37], and Li et al. [38]. Both heat conduction and viscoelasticity are included in these models along with the fractional order effects. Still, the volumetric properties of the material have been disregarded in relation to the relaxing effects.

In the field of thermo-viscoelasticity [39-42], the integral form of the equations demonstrates a close relationship between stress, strain, and temperature. Stress is represented as a function that changes over time and considers the impact of strain and temperature using relaxation functions. The relaxation function describes the time-dependent alteration in the mechanical behavior of materials caused by strain and temperature. It enables the determination of stress distribution by considering the past strain and thermal fluctuations. By subjecting the thermo-viscoelastic material to complicated thermal and mechanical loads, it is possible to accurately anticipate its entire response. Atangana and Baleanu [43] maintained "The non-singular kernel in integral operator will be helpful to discuss real world problems and it also will have a great advantage when using the Laplace transform to solve some physical problems with initial conditions and good for describing the dynamics of systems with memory effect for small and large times". Using the Mittag-Leffler function [44] instead of the exponential relaxation function in mathematical models provides a more precise and comprehensive depiction of the behavior of thermo-viscoelastic materials. The conventional exponential function is employed to depict the process of relaxation in materials, elucidating the decline of stress over time in a swift and precise manner. This behavior is characteristic of materials that exhibit a straightforward linear viscoelastic response. Conversely, the Mittag-Leffler function extends the exponential function and can depict more intricate behaviors. It offers versatility in modelling the decay pattern, enabling the inclusion of long-lasting effects that do not diminish as rapidly as in the exponential model. This more accurately represents the behavior of actual materials, which may have extended time delays and more intricate interactions. The influence of the fractional order derivative α on the behaviour of viscoelastic materials, especially in material sciences, biological tissues, Geological materials and engineering contexts, is well illustrated in daily real-life applications [45, 46]. Within the domain of polymers and composites, fractional analysis provides a precise description of intricate viscoelastic phenomena. The value of this parameter determines the varied responses patterns the materials exhibit under various thermomechanical effects. Polymers with long-term memory effects, such as those with prolonged creep under a constant load, exhibit low values of α, leading to a gradual accumulation of deformation over time. On the other hand, polymers that quickly respond to stress and rapidly stabilize have significant α values, which indicate a shorter memory and behavior that closely mimic thermoelastic theory. In biological tissues, such as skin and muscles, scarred or ageing tissues maintain their deformation for extended periods of time when the load is removed, which is indicated by lower α values. Conversely, healthy tissues such as youthful muscles rapidly recover their form, which indicates high α values. Geological materials that display a progressive and uninterrupted response to stress, such as the slow displacement of rocks caused by the force of mountains, are associated with lower α values. Moreover, geological materials that exhibit rapid responses to stress variations, such as sandy soils under varying thermomechanical effects, have high α values. Understanding the influence of α is crucial for developing new materials with specific mechanical properties, analysing structural failures in civil engineering.

Recently, Sherief et al. [47, 48] proposed a novel approach to the study of generalized fractional hereditary thermoelasticity that includes the Mittag-Leffler relaxation function. To demonstrate the results, we resolved a one-dimensional thermo-viscoelastic problem involving the cylinder cavity. The governing equations of our new model are analytically solved using a Laplace transform. The solution in the transformed domain is obtained, and then the Laplace transform is numerically inverted using a Fourier expansion technique [49-52]. In this process, we have used PMMA as the viscoelastic material and copper as the elastic material. Ultimately, based on the numerical results and accompanying graphics, a conclusion has been obtained on the thermo-viscoelasticity model.

2. The governing equation of the thermoviscoelastic model [47, 48]

We study a continuous viscoelastic medium enclosed within a volume V and bounded by a closed surface S. Denote a point's position vector as $\boldsymbol{x}(x_1, x_2, x_3)$. It is subject to a heat source of strength Q per unit mass and a body force F_i per unit mass. The strain tensor components e_{ij}, which are defined at each point of the body by

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1}$$

where $e_{ij} = e_{ji}$, $e = e_{ii} = u_{i,i}$ is the cubical dilation and u_i are the components of the displacement vector. The vector's component in the $x_i -$ direction is denoted by the subscript i and the standard notations: $(\cdot)_{,i} = (\partial/\partial x_i)(\cdot)$, where x_i is the system coordinates.

For a linear thermoviscoelastic material, we consider the stress tensor components $\sigma_{ij}(\boldsymbol{x},t)$, which are related to $e_{ij}(\boldsymbol{x},t)$ and temperature $\theta(\boldsymbol{x},t)$ by a convolution integral as follows:

$$\sigma_{ij}(\boldsymbol{x},t) = \tau^\alpha \int_0^t C_{ijkl} E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\upsilon)^\alpha\right)\frac{\partial e_{kl}(x,\upsilon)}{\partial \upsilon}d\upsilon - \alpha_T \tau^\alpha \int_0^t \gamma_{ij} E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\upsilon)^\alpha\right)\frac{\partial \theta(x,\upsilon)}{\partial \upsilon}d\upsilon, \tag{2}$$

where the variable θ is defined as the difference between the absolute temperature T and the reference temperature ϑ_0. It is required that the absolute value of this difference divided by the reference temperature, $\left|\frac{T-\vartheta_0}{\vartheta_0}\right|$, is much smaller than 1, α_T is the coefficient of linear thermal expansion, τ is a positive time constant for the ratio of the shear viscosity to Young's modulus, E_α represents the relaxation function related to the mechanical stress-strain response, $0 < \alpha < 1$, is the fractional order parameter characterizing the material's viscoelastic behavior, C_{ijkl} and γ_{ij} are both tensorial functions of the material, with C_{ijkl} being a fourth-order tensor and γ_{ij} being a second-order tensor. Furthermore, it is presumed that the subsequent symmetry relations are valid:

$$\left(C_{ijkl} = C_{klij} = C_{ikjl} = C_{jlik},\ \gamma_{ij} = \gamma_{ji}\right).$$

By substituting the expression from Equation (2) into the equation of motion, which is in the given form,

$$\sigma_{ji,j} + \rho F_i = \rho \ddot{u}_i, \tag{3}$$

Here, ρ represents the density, which is independent of time t and F_i is the vector component of the body force. The dot symbol indicates differentiation with respect to time. Therefore, we acquire

$$\rho\ddot{u}_i = \rho F_i + \tau^\alpha \int_0^t C_{ijkl} E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\upsilon)^\alpha\right)\frac{\partial e_{kl,j}(x,\upsilon)}{\partial \upsilon}d\upsilon - \alpha_T \tau^\alpha \int_0^t \gamma_{ij} E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\upsilon)^\alpha\right)\frac{\partial \theta_{,j}(x,\upsilon)}{\partial \upsilon}d\upsilon. \tag{4}$$

We can express the equation representing the balance of entropy density S as follows:

$$-q_{i,i} + \rho Q = \rho\vartheta_0 \frac{\partial S}{\partial t}, \tag{5}$$

where q_i represents the heat flux vector, and S represents the entropy per unit mass.

Let us assume that a thermally conducting viscoelastic solid, which is experiencing minor strain and small temperature fluctuations [53], may be described by the following equation:

$$\rho\vartheta_0 S = \rho c_E \theta + \alpha_T \tau^\alpha \vartheta_0 \int_0^t \gamma_{ij} E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\upsilon)^\alpha\right)\frac{\partial e_{ij}(x,\upsilon)}{\partial \upsilon}d\upsilon, \tag{6}$$

where c_E is the specific heat at constant strain. We notice that, the constitutive equation (2) and the entropy balance equation (6) coincide on equations (3.8) and (3.9) in Ref. [54].

Hence, the energy balance equation is obtained from Eqs. (5) and (6), as

$$-q_{i,i} + \rho Q = \frac{\partial}{\partial t}\left(\rho c_E \theta + \alpha_T \tau^\alpha \vartheta_0 \int_0^t \gamma_{ij} E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\upsilon)^\alpha\right)\frac{\partial e_{ij}(x,\upsilon)}{\partial \upsilon}d\upsilon\right), \tag{7}$$

The above equations are supplemented by modified Fourier law [6, 55], namely

$$\left(1 + \tau_q \frac{\partial}{\partial t}\right) q_i(\boldsymbol{x},t) = -k_{ij}\theta_{,j}(\boldsymbol{x},t), \tag{8}$$

where $\tau_q \geq 0$ is the relaxation time and k_{ij} is a thermal conductivity.

Operate with div-operator on both sides for the previous equation and using Eq. (7), we obtain

$$\left(k_{ij}\theta_{,j}\right)_{,i} = \left(\frac{\partial}{\partial t} + \tau_q \frac{\partial^2}{\partial t^2}\right)\left(\rho c_E \theta + \alpha_T \tau^\alpha \vartheta_0 \int_0^t \gamma_{ij} E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\upsilon)^\alpha\right)\frac{\partial e_{ij}(x,\upsilon)}{\partial \upsilon}d\upsilon\right) - \rho\left(Q + \tau_q \frac{\partial Q}{\partial t}\right). \tag{9}$$

Here, all functions are dependent on both the variables $\boldsymbol{x}$ and t. The summation notation is employed while disregarding the micro rotations.

Assuming an isotropic body, the tensorial functions can be expressed as [41, 47]

$$C_{ijkl} = \left(\lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right) \tag{10}$$

$$\gamma_{ij} = (3\lambda + 2\mu)\delta_{ij},\ k_{ij} = k\delta_{ij}, \tag{11}$$

where λ, μ are Lame's constants, δ_{ij} is the Kronecker delta.

The integral terms in the right-hand side of the previous governing equation denote a non-local fractional derivative with a non-singular kernel, which was introduced by Atangana and Baleanu [43].

3. Formulating the problem

Let us examine a thermoviscoelastic solid that is homogeneous and has the same properties in all directions. These solids fills the space within a hollow circular cylinder that is indefinitely long. The cylinder's inner radius is a, and its outer radius is b, where $a < b$. A cylindrical coordinate system (r, φ, z) is used, with the z-axis serving as the cylinder's axis. It is presumed that the medium is initially in a state of rest and there are no external forces or sources of heat. Traction-free boundary surfaces of the cylinder are subject to a thermal shock disturbance that is time-dependent. Based on the physics involved, all the physical quantities depend solely on the variables r and t only. To solve the problem, we need obtain the radial displacement component u_r and the non-zero stress components σ_{ji} in the specified region. The displacement vector $\mathbf{u}$ is defined by its components $\langle u_r(r,t), 0, 0\rangle$. The governing equations for generalized isotropic thermal viscoelastic over the domain $(r, \varphi, z, t) \in [a,b] \times [0, 2\pi] \times (-\infty, \infty) \times [0, \infty)$ are as follows:

(i) The strain tensor includes non-vanishing components, as indicated by

$$e_{rr} = \frac{\partial u_r}{\partial r}, \tag{12}$$

$$e_{\varphi\varphi} = \frac{u_r}{r}, \tag{13}$$

and the cubical dilatation e is given by the following expression:

$$e = \operatorname{div} \mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r}. \tag{14}$$

(ii) The equation of motion, in the absence of any external forces, can be represented in vector form [13]

$$\rho \ddot{\mathbf{u}}(r,t) = \mu\tau^{\alpha} \int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial}{\partial v}\nabla^2 \mathbf{u}(r,v)dv + (\lambda+\mu)\tau^\alpha \int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial}{\partial v}\nabla e(r,v)dv - (3\lambda+2\mu)\tau^\alpha \alpha_T \int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial}{\partial v}\nabla\theta(r,v)dv, \tag{15}$$

where $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$ is Laplace's operator, where all functions depend on r, t.

An alternative expression for the equation mentioned above, using cubic dilatation e, may be derived by applying the divergence operator to both sides.

$$\rho \ddot{e}(r,t) = (\lambda+2\mu)\tau^\alpha \int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial}{\partial v}\nabla^2 e(r,v)dv - (3\lambda+2\mu)\tau^\alpha\alpha_T \int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial}{\partial v}\nabla^2\theta(r,v)dv, \tag{16}$$

(iii) The heat conduction equation without a heat source

$$k\nabla^2\theta = \left(\frac{\partial}{\partial t}+\tau_q\frac{\partial^2}{\partial t^2}\right)\left(\rho c_E\theta + (3\lambda+2\mu)\tau^\alpha\alpha_T\vartheta_0\int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial e}{\partial v}dv\right). \tag{17}$$

(iv) The non-vanishing components of the stress tensor are given by

$$\sigma_{rr}(r,t) = 2\mu\tau^\alpha\int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial}{\partial v}\left(\frac{\partial u_r}{\partial r}\right)dv + \lambda\tau^\alpha\int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial e}{\partial v}dv - (3\lambda+2\mu)\tau^\alpha\alpha_t\int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial\theta}{\partial v}dv. \tag{18}$$

$$\sigma_{\varphi\varphi}(r,t) = \lambda\tau^\alpha\int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial e}{\partial v}dv - (3\lambda+2\mu)\tau^\alpha\alpha_t\int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial\theta}{\partial v}dv. \tag{19}$$

$$\sigma_{zz}(r,t) = \lambda\tau^\alpha\int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial e}{\partial v}dv - (3\lambda+2\mu)\tau^\alpha\alpha_t, \tag{20}$$

associated with the homogeneous initial conditions and the thermal and mechanical boundary conditions can be expressed as follow:

$$\theta(a,t) = \frac{1}{\alpha_t}f_1(t),\ \theta(b,t) = \frac{1}{\alpha_t}f_2(t),\ t\geq 0 \tag{21}$$

$$\sigma_{rr}(a,t) = \sigma_{rr}(b,t) = 0,\ t \geq 0. \tag{22}$$

where $f_i(t), (i = 1,2)$ are non-dimensional known functions, which defined later.

We will utilize the subsequent dimensionless variables.

$$(r^*, u_r^*) = \ell(r,u_r)\ ,(t^*,\tau_q^*) = \ell v\,(t,\tau_q), \theta^* = \alpha_t\theta, \sigma_{ij}^* = \frac{\sigma_{ij}}{\mu} \tag{23}$$

where $\ell = \frac{\rho c_E v}{k}$, and $v = \sqrt{(2\mu+\lambda)/\rho}$, is the speed of propagation of isothermal elastic waves.

By using the non-dimensional variables mentioned before, the governing equations (16)-(20) may be expressed without the asterisks for the intent of simplicity.

$$\ddot{e}(r,t) = \hat{\mathcal{G}}_\alpha(\nabla^2 e - \nabla^2\theta), \tag{24}$$

$$\ddot{u}_r = \hat{\mathcal{G}}_\alpha\left(\frac{\partial e}{\partial r} - \frac{\partial\theta}{\partial r}\right), \tag{25}$$

$$\nabla^2\theta = \left(\frac{\partial}{\partial t}+\tau_q\frac{\partial^2}{\partial t^2}\right)\left(\theta + \varepsilon\,\hat{\mathcal{G}}_\alpha(e)\right). \tag{26}$$

$$\sigma_{rr} = \hat{\mathcal{G}}_\alpha\left(2\frac{\partial u_r}{\partial r} + (\xi^2-2)e - \xi^2\theta\right) \tag{27}$$

$$\sigma_{\varphi\varphi} = \hat{\mathcal{G}}_\alpha\left(\frac{2u_r}{r} + (\xi^2-2)e - \xi^2\theta\right) \tag{28}$$

$$\sigma_{zz} = \hat{\mathcal{G}}_\alpha\left((\xi^2-2)e - \xi^2\theta\right) \tag{29}$$

where $\varepsilon = \frac{\alpha_t^2(2\mu+3\lambda)^2\vartheta_0}{\rho c_E}$, $\xi^2 = \frac{2\mu+\lambda}{\mu}$, the operator $\hat{\mathcal{G}}_\alpha(\cdot)$ is defined for any function $h(r,t)$ of class $\mathrm{H}^1(a_0,a_1), a_0 < a_1$ as follows:

$$\hat{\mathcal{G}}_\alpha(h(r,t)) = \tau^\alpha\int_0^t E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-v)^\alpha\right)\frac{\partial h(r,v)}{\partial v}dv.$$

4. Solution in the transform domain

We shall now define the Laplace transform (denoted by a bar) with respect to a function $h(r,t)$ by the relation [56]

$$\mathcal{L}\{h(r,t);t\} = \int_0^\infty h(r,t)e^{-st}\,dt = \bar{h}(r,s), \tag{30}$$

where $h(r,t)$ is continuous function on time, s is the Laplace parameter.

Applying the Laplace transform to both sides of equations (*24*)-(*29*), and using the homogenous initial conditions, we arrive at:

$$\left(\nabla^2 - \frac{s^2}{\varpi}\right)\bar{e} - \nabla^2\bar{\theta} = 0, \tag{31}$$

$$\frac{s^2}{\varpi}\bar{u}_r = \frac{\partial\bar{e}}{\partial r} - \frac{\partial\bar{\theta}}{\partial r}, \tag{32}$$

$$\left(\nabla^2 - \left(s+\tau_q s^2\right)\right)\bar{\theta} - \varepsilon\varpi\left(s+\tau_q s^2\right)\bar{e} = 0, \tag{33}$$

$$\bar{\sigma}_{rr} = \varpi\left(2\frac{\partial\bar{u}_r}{\partial r} + (\xi^2-2)\bar{e} - \xi^2\bar{\theta}\right), \tag{34}$$

$$\bar{\sigma}_{\varphi\varphi} = \varpi\left(\frac{2\bar{u}_r}{r} + (\xi^2-2)\bar{e} - \xi^2\bar{\theta}\right), \tag{35}$$

$$\bar{\sigma}_{zz} = \varpi\left((\xi^2-2)\bar{e} - \xi^2\bar{\theta}\right), \tag{36}$$

where $\mathcal{L}\{\hat{\mathcal{G}}_\alpha(h(r,v))\} = \varpi\bar{h}(r,s)$, and $\varpi = \frac{s^\alpha}{s^\alpha + \left(\frac{\alpha}{1-\alpha}\right)}$

The transformed non dimensional boundary conditions (*21*) and (*22*) become

$$\bar{\theta}(a,s) = \bar{f}_1(s),\ \bar{\theta}(b,s) = \bar{f}_2(s), \tag{37}$$

$$\bar{\sigma}_{rr}(a,s) = \bar{\sigma}_{rr}(b,s) = 0, \tag{39}$$

Eliminating $\bar{\theta}$ between equations (*31*) and (*33*), we obtain

$$\left(\nabla^4 - \nabla^2\left(\frac{s^2}{\varpi} + (s+\tau_q s^2)(1+\varepsilon\varpi)\right) + \frac{s^3}{\varpi}(1+\tau_q s)\right)\bar{e}(r,s) = 0 \tag{39}$$

The above equation can be factorized as

$$\prod_{i=1}^{2}(\nabla^2 - k_i^2)\,\bar{e}(r,s) = 0 \tag{40}$$

where $k_i^2, i = 1,2$ are the roots with positive real parts of the characteristic equation

$$k^4 - k^2\left(\frac{s^2}{\varpi} + \left(s+\tau_q s^2\right)(1+\varepsilon\varpi)\right) + \frac{s^3}{\varpi}\left(1+\tau_q s\right) = 0. \tag{41}$$

Hence, the solutions of equation (42) is given by

$$\bar{e}(r,s) = \sum_{i=1}^{2} k_i^2\left(A_i I_0(k_i r) + B_i K_0(k_i r)\right), \tag{42}$$

where $A_i, B_i, (i = 1,2)$ are parameters depending on s. $I_n(r)$ and $K_n(r)$ are the modified Bessel functions of the first and second kinds of order n, respectively.

Similarly, eliminating $\bar{e}(r,s)$, between equations (*31*) and (*33*), we obtain

$$\left(\nabla^4 - \nabla^2\left(\frac{s^2}{\varpi} + (s + \tau_q s^2)(1 + \varepsilon\varpi)\right) + \frac{s^3}{\varpi}(1 + \tau_q s)\right)\bar{\theta}(r,s) = 0 \qquad (45)$$

For $(a \le r \le b)$, the solution of equation (45) is

$$\bar{\theta}(r,s) = \sum_{i=1}^{2}\left(k_i^2 - \frac{s^2}{\varpi}\right)\,(A_i' I_0(k_i\, r) + B_i' K_0(k_i r)), \qquad (46)$$

where $A_{in}', B_{in}', i = 1,2$ are parameters depending on s.

Substituting from equations (44) and (46) into the equation (*31*), we obtain

$A_i'(s) = A_i(s),\ B_i'(s) = B_i(s),\ \forall\, i = 1,2.$

Substituting into equation (46), we get

$$\bar{\theta}(r,s) = \sum_{i=1}^{2}\left(k_i^2 - \frac{s^2}{\varpi}\right)\,(A_i I_0(k_i r) + B_i K_0(k_i r)), \qquad (47)$$

By applying the Laplace transform to equation (14), and using Eq. (44), we obtain

$$\frac{\partial(r u_r)}{\partial r} = r\sum_{i=1}^{2} k_i^2 (A_i I_0(k_i r) + B_i K_0(k_i r)). \qquad (48)$$

integrating both sides of Eq. (48) with respect to r, and using the given relation of the modified Bessel functions of the first kind [23,24]

$$\int r I_0(r)dr = r I_1(r), \text{ and } \int r K_0(r)dr = -r K_1(r).$$

Hence, the bounded solution $\bar{u}_r$ for $a \le r \le b$, is given by

$$\bar{u}_r(r,s) = \sum_{i=1}^{2} k_i (A_i I_1(k_i r) - B_i K_1(k_i r)), \qquad (47)$$

Substituting from equation (44), (47), and (47) into equations (34) and (35), we obtain the stress components, such that

$$\bar{\sigma}_{rr}(r,s) = \frac{\varpi}{r}\sum_{i=1}^{n}(A_i[\zeta^2 r I_0(k_i r) - 2k_i I_1(k_i r)] + B_i[\zeta^2 r K_0(k_i r) + 2k_i K_1(k_i r)]), \qquad (48)$$

$$\bar{\sigma}_{\varphi\varphi}(r,s) = \frac{1}{r}\sum_{i=1}^{n}\left(A_{in}\left[(\zeta^2 - 2k_i^2) r I_0(k_i r) + 2k_i I_1(k_i r)\right] + B_{in}\left[(\zeta^2 - 2k_i^2) r K_0(k_i r) + 2k_i K_1(k_i r)\right]\right) \qquad (49)$$

where $\zeta^2 = \frac{\xi^2 s^2}{\varpi}$.

5. Application

Without loss of generality, the boundary thermal load $f_1(t) = f_2(t) = f(t)$ as a time dependence thermal shock, we obtain

$$f(t) = H(t). \qquad (50)$$

where $H(t)$ is the Heaviside unit step function. Taking the Laplace transform of the previous equation, we obtain

$$\bar{f}(s) = \frac{1}{s}. \qquad (51)$$

Equations (*38*), (*40*), (47), (48), and (*50*) immediately give the system of four linear equations in the unknown parameters $A_i(s), and B_i(s), i = 1,2.$

$$\sum_{i=1}^{2}\left(k_i^2 - \frac{s^2}{\varpi}\right)\,(A_i I_0(k_i a) + B_i K_0(k_i a)) = \frac{1}{s},$$

$$\sum_{i=1}^{2}\left(k_i^2 - \frac{s^2}{\varpi}\right)\,(A_i I_0(k_i b) + B_i K_0(k_i b)) = \frac{1}{s},$$

$$\sum_{i=1}^{n}(A_i[\zeta^2 a I_0(k_i a) - 2k_i I_1(k_i a)] + B_i[\zeta^2 a K_0(k_i r) + 2k_i K_1(k_i a)]) = 0,$$

$$\sum_{i=1}^{n}(A_i[\zeta^2 b I_0(k_i b) - 2k_i I_1(k_i b)] + B_i[\zeta^2 b K_0(k_i r) + 2k_i K_1(k_i b)]) = 0.$$

6. Numerical Results

In order to invert the Laplace transform in the above equations (47)- (49), we adopt a numerical inversion method based on a Fourier series expansion [51]. The numerical inversion method used to find the solution in the physical domain are listed in [57-59]. The numerical code has been prepared using Fortran programming language. The accuracy maintained was six digits for the numerical program, and the numerical results are illustrated graphically using MATLAB's robust graphical features. Numerical experiments were conducted at three instant values of time (small value of time, namely $t = 0.1, 0.14$ and large value of time, namely $t = 0.5$) at different value of $\alpha \in (0,1)$.

The fractional order parameter α expresses memory or relaxation in thermoviscoelastic materials. This parameter affects the material's response to mechanical and thermal changes; a lower value of α means that the material keeps a considerable degree of memory from its prior states. The material then shows very relaxed behavior and reacts slowly to thermomechanical changes. Conversely, when $\alpha \to 1$, the material exhibits more normal viscoelastic behavior and has less memory retention. Here, the material depends less on its deformation history and reacts to thermomechanical changes as quickly as an exponential function.

Let us now determine the inverse transforms for the case of small values of time $t \to 0$. By the initial value theorem of the Laplace transforms [60, 61], this corresponds to large values of s. Taking $x = s^{-1}$, (x is small) and expanding the roots k_1^2 and k_2^2 of the characteristic equation (43) into Maclaurin's series, we obtain

$$k_i^2 = s^2(b_{i0} + O(x^\alpha)),\ (i = 1,2). \qquad (52)$$

where $b_{i0} = \frac{1}{2}\left((\tau_0 + \tau^{-\alpha} + \varepsilon\tau_0\tau^\alpha) + (-1)^{i+1}\sqrt{\varepsilon\tau_0^2\tau^\alpha(\varepsilon\tau^\alpha + 2) + \tau_0(2\varepsilon + \tau_0) - 2\tau_0\tau^{-\alpha} + \tau^{-2\alpha}}\right),\ (i = 1,2).$

Taking square roots of equation (51) and expanding again, we get

$$k_i = s\left(\sqrt{b_{i0}} + O(x^\alpha)\right),\ \frac{1}{\sqrt{k_i}} = \frac{1}{\sqrt{s}}\left(b_{i0}^{-1/4} + O(x^\alpha)\right)\ (i = 1,2). \qquad (53)$$

Hence,

$$\left(k_i^2 - \frac{s^2}{\varpi}\right) = s^2\left((b_{i0} - \tau^\alpha) + O(x^\alpha)\right), i = 1,2. \qquad (54)$$

Since $k_i = O(s), i = 1,2$ and k_i is large. Thus, the modified Bessel functions of the first and second kinds have asymptotic expansions when the argument $k_i r$ is large [62].

$$I_0(k_i r) \cong \frac{e^{k_i r}}{\sqrt{2\pi k_i r}}\ ,\ K_0(k_i r) \cong \sqrt{\frac{\pi}{2k_i r}}\, e^{-k_i r},\ i = 1,2\ ,\ r > 0 \qquad (55)$$

By solving the linear system to determine $A_i, B_i, i = 1,2$, using the relations (54) and expanding into Maclaurin's series with the help of the MATLAB symbolic toolbox, we obtain

$$A_i = (-1)^i \frac{e^{-b\sqrt{b_{i0}}s}}{\sqrt{s^5}}\left(\frac{\sqrt[4]{b_{i0}}}{b_{20} - b_{10}}\sqrt{2\pi b} + O(x^\alpha)\right),\ B_i = (-1)^i \frac{e^{(2a-b)\sqrt{b_{i0}}s}}{\sqrt{s^5}}\left(\frac{\sqrt[4]{b_{i0}}}{b_{10} - b_{20}}\sqrt{\frac{2b}{\pi}} + O(x^\alpha)\right),\ (i = 1,2). \qquad (56)$$

Substituting from (52), (53) and (54) into (47), we obtain

$$\bar{\theta}(r,s) = \frac{1}{s}\sqrt{\frac{b}{r}}\sum_{i=1}^{2}(-1)^i\left(\frac{(b_{i0} - \tau^\alpha)}{b_{20} - b_{10}} + O(x^\alpha)\right)\left(e^{s[(r-b)\sqrt{b_{i0}} + O(x^\alpha)]} + e^{s[(-r+2a-b)\sqrt{b_{i0}} + + O(x^\alpha)]}\right). \qquad (55)$$

Now, we will use the Boley theorem [61, 63, 64], which is highly valuable for determining the wave front location and speeds of thermal and mechanical waves in Laplace transform expressions. This theorem is particularly beneficial when these expressions involve exponential functions in their simplest form, up to order $O(x^{\alpha})$.

The temperature distribution exhibits two interrelated waves. In Equation (55), the four exponential functions correspond to two distinct types of waves that originate from the outer and inner boundary surfaces, specifically mechanical and thermal waves. Thus, the terms corresponding to the first and third terms represent mechanical waves, whereas the terms corresponding to the second and fourth terms indicate thermal waves. These waves are travelling from the outer and inner boundary surfaces of the cylinder cavity with finite wave speeds equal to $\left(\frac{1}{\sqrt{b_{i0}}}, i = 1,2\right)$. Additionally, the velocities of wave propagation for both mechanical and thermal waves depend explicitly on the three material parameters $(\varepsilon, \tau_0, \tau)$, which remain constant for the same material, and a new fractional parameter α. For any fixed value of fractional parameter α, the temperature distribution has a finite discontinuity across mechanical and thermal waves at the four wave fronts, as follows:

$$r_i|_{for \text{ waves originating at outer surface}} = b - \frac{t}{\sqrt{b_{i0}}}, i = 1,2,$$

$$r_i|_{for \text{ waves originating at inner surface}} = b - 2a + \frac{t}{\sqrt{b_{i0}}}, i = 1,2.$$

Furthermore, the radial stress component exhibits the same features as the temperature distribution mentioned before. Nevertheless, the distribution of radial displacement remains continuous at these places, but its first derivatives exhibit discontinuous behavior with a finite jump. The numerical calculation of all physical quantities and non-dimensional speeds for the mechanical and thermal wave in poly-methyl-methacrylate (PMMA) were computed based on the provided material constants in (Table 1).

The impact of the fractional parameter α on the speed of the mechanical and thermal waves can be observed in

Table **2**. An important observation is that as the fractional order parameter α increases, the speed of both thermal and mechanical waves decreases. The wave speeds at α = 0 are identical to those reported in Ref. [67, 68] for thermoelasticity with a single relaxation time.

(**Figure 1**) depicts the variation of dimensionless temperature $\theta(r,t)$ against the radial direction with varying values of α. It is seen that for any given value of α, the temperature distribution has a maximum value at the boundaries. This observation is consistent with the specified thermal boundary condition, which subjects the outside and inner boundary surfaces of the cylinder cavity to a thermal shock. Within the medium, the temperature gradually decays radially. For small values of time (e.g., $t = 0.1$), the sharp temperature decreases to zero because of the impact on the thermal wave front. For a longer duration of time (e.g., $t = 0.14$), the two thermal waves from both surfaces of cylinder cavity converge to $r = 2.0$, as this time is sufficient for both outer and inner thermal waves to reach to this location. When we increase the time (t = 0.5, for instance), we observe that the two thermal waves transferred from both boundary surfaces merge and blend with each other, leading to an overall increase in the temperature of the cylinder cavity. This behavior corresponds to the fact that the wave travels at a finite speed. Our calculations, with an accuracy of 10^{-6}, indicate that the temperature profile distribution values are almost identical for various values of α. The model is not limited by the fact that the parameter α has no direct impact on temperature. Temperature is the measurement used to quantify the amount of thermal energy present in a system. The parameter α governs the response of a material to changes in temperature and affects its stress and strain characteristics.

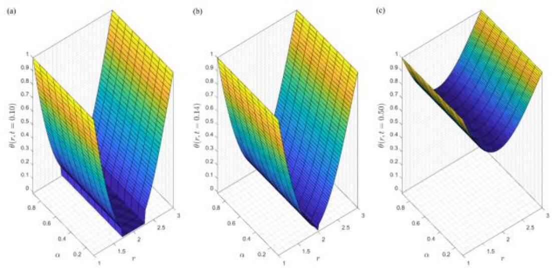

Figure 1. Dimensionless temperature vs. distance at different values of α.

(**Figure 2**) shows the variation of the radial displacement, $u_r(r,t)$, against the radial direction for various values of α. It is observed that, for a given value of α, the absolute value of the displacement component u_r records a maximum value at the boundaries of the cylinder cavity and decreases gradually with propagation inside the medium to attain minimum values. Furthermore, the amplitude of the radial displacement diminishes uniformly across the entire domain as the value of α increases and approaches zero at a faster rate (i.e., for larger values of α, the rate of decay accelerates). Investigating the impact of the fractional parameter α on the displacement in thermoviscoelastic materials represented by the Mittag-Leffler function, small values of α allow the system to keep more memory. Accordingly, the system is still influenced by the earlier thermomechanical effects for a considerable amount of time. We can see the system as a long-term accumulation of little effects. The long-term memory consequences cause displacement to increase gradually. The displacement is little at first and keeps growing steadily. Thus, the system needs longer time to reach a substantial displacement. Displacement increases gradually initially but then keeps rising. With time, this ongoing buildup causes a considerable displacement. The material deforms gradually and more because of the continuing consequences of previous occurrences. A larger α means that the system keeps a smaller memory. Because past effects of stress and strain fade more quickly, the system reacts to current variations in stress and strain more swiftly. The displacement begins to increase more rapidly due to the system's shorter memory. Displacement becomes a steady state more quickly, even though it grows rapidly at first. Thermomechanical effects fade fast, and displacement stabilizes at a specific point very quickly. The system does not hold memory for extraordinarily long; as a result, the final displacement is less than in systems with a smaller α.

Table 1. Mechanical and thermal properties of PMMA at reference temperature $\vartheta_0 = 293^0 K$ [65, 66].

Property	λ	μ	k	c_E	ρ	α_T
Value	4.0×10^9	1.9×10^9	190	1470	1160	6.3×10^{-5}
Unit	$kg/m\,sec^2$	$kg/m\,sec^2$	$W/(mK)$	$J/(kgK)$	kg/m^3	K^{-1}

Table 2 Effect of fractional parameter α on waves speed [69].

α	Speed for Mechanical wave	Speed for thermal wave
0.9	0.1258925395	7.071067811
0.7	0.1995262147	7.071068395
0.5	0.3162275981	7.071071567
0.3	0.5011855488	7.071091583
0.1	0.7943112567	7.071218954
0.0	0.9999459087	7.071450317

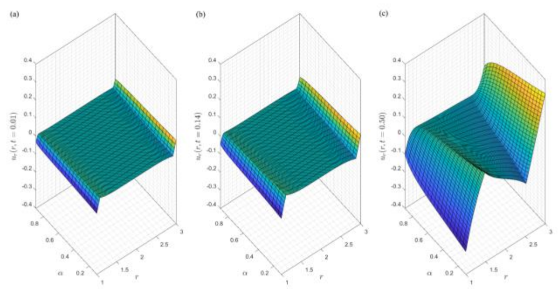

Figure 2. Dimensionless radial displacement vs. distance at different values of α.

(**Figure 3**) illustrates the variation of the radial stress component $\sigma_{rr}(r,t)$ against the radial direction at different values of α. For a given α value, the dimensionless radial stress component σ_{rr} records zero at the boundaries of the cylinder cavity, which agrees with the prescribed mechanical boundary condition that the outer and inner surfaces of the cylinder cavity are traction-free. (**Figure 3a**) depicts the radial stress component σ_{rr}, which begins at 0 and gradually rises to its maximum positive value until reaching the mechanical wave front. Subsequently, it descends to a negative value and continues to rise (although remaining negative) until it reaches the thermal wave front, at which point it abruptly transitions to zero. This behavior is replicated in (**Figure 3b**), at $t = 0.14$ as the two transferred thermal waves merge, as observed in (**Figure 1b**). In (**Figure 3c**) at $t = 0.5$, the gap in the middle is filled with waves originating from the outside and inner surfaces of the cylinder cavity, for more details, see Video S1. Video S1 displayed the variation in nondimensional temperature, radial displacement, and radial stress component against the radial direction. Examining the impact of the fractional parameter on the radial stress component reveals significant behavioral changes in the material. The system's long-term memory characteristics are shown for small values of α. Accordingly, because of the long-lasting impact of earlier thermomechanical effects, stress is increasing gradually and persisting over time. Slowly but surely, stress grows. When α is close to one, however, the system exhibits minimal memory properties, indicating that present or recent strain is the main factor affecting stress. Early thermomechanical accumulation results in reduced long-term stress, which stabilizes quickly. A thorough knowledge of material dynamics is essential to managing stress in many thermomechanical problems.

In all these figures, the decrease of α results an increase in the magnitude of the radial stress component. It is important to note that a change in the fractional parameter significantly impacts mechanical functions such as displacement and stress, while having a negligible effect on temperature, as illustrated in (**Figure 4**). The effect is significant in terms of the location of the front and the speed of the thermal wave. At small values of time, all the curves exhibit the second sound effect.

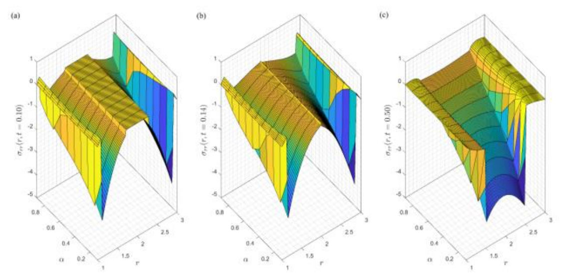

Figure 3. Dimensionless radial stress component vs. distance at different values of α.

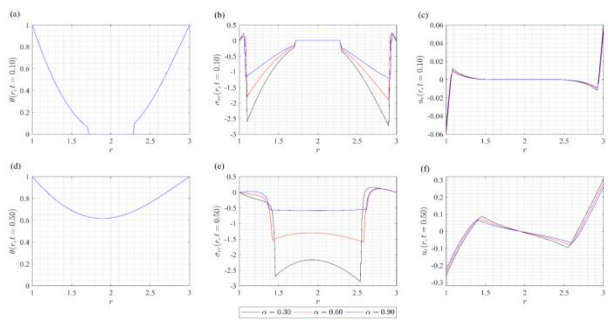

Figure 4. Effect of fractional parameter α on some physical variables.

It is observed from (**Figure 4**) that, for any fixed values of the fractional parameter, the profiles of temperature, radial stress component, and radial displacement exhibit asymmetry around $r = 2$. This asymmetry arises due to the variation in temperature distribution between the inner surface of the cylindrical cavity at $r = 1$ and the outer surface at $r = 3$. The inner surface has a smaller area compared to the outer surface, resulting in a different temperature profile, and consequently affecting the amplitude of all physical quantity profiles. Nonetheless, the positions of the thermal and mechanical waves remain well-defined and confined within the medium.

The physics of the problem allows us to identify the deformation arising solely from radial displacement. Therefore, we can select any cross section of the cylinder cavity and analyze its deformation by calculating the changes in area at various instantaneous times. We numerically compute the surface area of a deformed grid using the provided MATLAB toolbox. This approach builds a mesh grid using given radial $(1 \leq r \leq 3)$ and angular $(0 \leq \varphi \leq 2\pi)$ coordinates, then converts them to Cartesian coordinates. Using the results of the radial displacement, we next modify the Cartesian coordinates to represent the deformation. We determine the surface area by triangulating [70] and adding up the areas of each triangle formed by nearby grid points, as illustrated in (**Figure 5**).

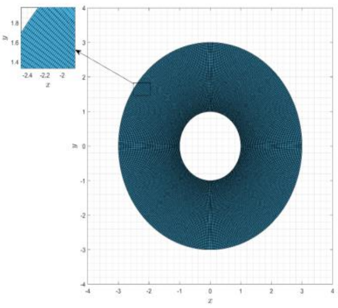

Figure 5. Deformed cross-section with finer mesh density.

To validate this approach, we used finer grids to calculate the area of the first known shape of the cross section of the cylinder cavity. This yielded a value of $A_0 = 25.132$, which coincides with the exact values up to three digits. In (**Figures 6-7**), we demonstrate the significant impact of the fractional order parameter on the deformation cross section at two specific moments in time, $t = 0.07$ and $t = 0.1$. We also calculate the area change, δA, which is equal to $A(r,\alpha,t) - A_0$. At any fixed instant value of time, the decreasing value of the fractional order parameter leads to an increase in the cross-section area of the cylinder cavity. Specifically, the small values of the fractional order parameter cause more deformation in the cylinder cavity, indicating a longer memory time. We conducted a comprehensive analysis spanning the entire duration. Video S2 shows the deformation in the cylinder cavity's cross-section.

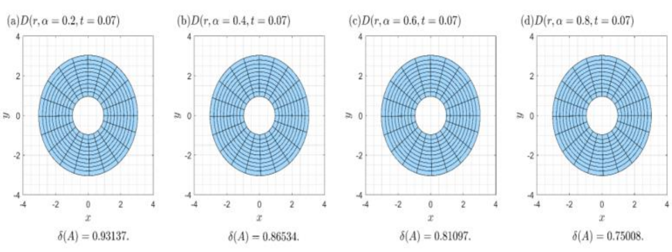

Figure 6. Effect of fractional parameter α on the deformatio cross-section and its area with reduced mesh density at $t = 0.07$.

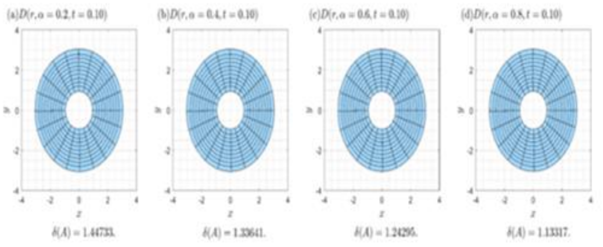

Figure 7. . Effect of fractional parameter α on the deformation of cross-section and its area with reduced mesh density at $t = 0.10$.

Conclusion

The main goal of our paper is to solve one dimensional thermoviscoelastic problem of cylinder cavity under a new fractional mathematical model of generalized thermoviscoelasticity associated with non-singular fractional relaxation function. Unlike the other previous models in where the fractional parameter α appears only in the equation of heat conduction. The solution of the governing equations of this model behaves in an equivalent manner to the solution of generalized thermoviscoelasticity model though with some different values. We suggest classifying the different material according to the value of the new fractional parameter α. A very important difference is that the presented model predicts finite speeds of propagation for thermal and mechanical waves while the speed of thermal wave in all previous fractional models in both fields of thermoelasticity and thermoviscoelasticity [36, 71, 72] is infinite contrary to physical observations. Our study's primary limitation is its limited scope of experimental validation. While our study demonstrates strong numerical validation, it is essential to conduct appropriate real-world experimental validation for the suggested models. Also, accurate determination of α is essential for diverse materials and physical problems since even minor variations in this value can cause significant changes in the thermomechanical responses. Nowadays, the knowledge (***Generalized thermoviscoelasticity theory Associated with non-singular fractional Relaxation function***) can be utilized by engineers, more particularly by mechanical and structural engineers for designing machine elements like synthesize metal.

Supplementary materials

Video S1: The variation in nondimensional temperature, radial displacement, and radial stress component against r at different values of α through the time interval.

Video S2: Deformed cross-section with reduced mesh density at different values of α through the time interval.

Conflict of interest

The authors declare that they have no Conflict of interest.

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References

[1] Tschoegl, N. W. Time Dependence in Material Properties: An Overview. Mechanics of Time-Dependent Materials 1997, 1 (1).

[2] Grün, F. Mathematical structure of the theories of viscoelasticity. BERNHARD GROSS Hermann & Cie, Paris, 1953. 74 pp., 600 francs. Wiley Online Library: 1954.

[3] Atkinson, C.; Craster, R. Theoretical aspects of fracture mechanics. Progress in Aerospace Sciences 1995, 31 (1), 1-83.

[4] Rajagopal, K.; Saccomandi, G. On the dynamics of non-linear viscoelastic solids with material moduli that depend upon pressure. International journal of engineering science 2007, 45 (1), 41-54.

[5] Liu, I.-S. On Fourier's law of heat conduction. Continuum mechanics and Thermodynamics 1990, 2, 301-305.

[6] Lord, H. W.; Shulman, Y. A generalized dynamical theory of thermoelasticity. Journal of the Mechanics and Physics of Solids 1967, 15 (5), 299-309.

[7] Sherief, H. H.; Anwar, M. N. A problem in generalized thermoelasticity for an infinitely long annular cylinder. International journal of engineering science 1988, 26 (3), 301-306.

[8] Sherief, H. H.; Raslan, W. 2D problem for a long cylinder in the fractional theory of thermoelasticity. Latin American Journal of Solids and Structures 2016, 13, 1596-1613.

[9] Sherief, H. H.; El-Maghraby, N. M. Effect of body forces on a 2D generalized thermoelastic long cylinder. Computers & Mathematics with Applications 2013, 66 (7), 1181-1191.

[10] Sherief, H. H.; Hamza, F. A. Modeling of variable thermal conductivity in a generalized thermoelastic infinitely long hollow cylinder. Meccanica 2016, 51, 551-558.

[11] Kovalenko, A.; Karnaukhov, V. A linearized theory of thermoviscoelasticity. Polymer Mechanics 1972, 8 (2), 194-199.

[12] El-Karamany, A. S.; Ezzat, M. A. Thermal shock problem in generalized thermo-viscoelasticty under four theories. International Journal of Engineering Science 2004, 42 (7), 649-671.

[13] Ezzat, M. A.; Othman, M. I.; El-Karamany, A. S. State space approach to generalized thermo-viscoelasticity with two relaxation times. International journal of engineering science 2002, 40 (3), 283-302.

[14] Ezzat, M. A.; El-Karamany, A. S. The uniqueness and reciprocity theorems for generalized thermo-viscoelasticity with two relaxation times. International journal of engineering science 2002, 40 (11), 1275-1284.

[15] El-Karamany, A. S.; Ezzat, M. A. Discontinuities in generalized thermo-viscoelasticity under four theories. Journal of Thermal Stresses 2004, 27 (12), 1187-1212.

[16] El-Karamany, A. S.; Ezzat, M. A. On the boundary integral formulation of thermo-viscoelasticity theory. International journal of engineering science 2002, 40 (17), 1943-1956.

[17] El-Karamany, A. S.; Ezzat, M. A. Uniqueness and reciprocal theorems in linear micropolar electro-magnetic thermoelasticity with two relaxation times. Mechanics of Time-Dependent Materials 2009, 13, 93-115.

[18] Ezzat, M. A. The relaxation effects of the volume properties of electrically conducting viscoelastic material. Materials Science and Engineering: B 2006, 130 (1-3), 11-23.

[19] Ezzat, M.; Awad, E. A Theory of Heat and Mass Transfer in Viscoelastic Solids with Microstructures. Journal of Thermal Stresses 2011, 34 (8), 795-816.

[20] Caputo, M.; Mainardi, F. A new dissipation model based on memory mechanism. Pure and applied Geophysics 1971, 91, 134-147.

[21] Di Fisica dell'Universith-Bologna, I. Linear Models of Dissipation in Anelastic Solids.

[22] Caputo, M. Vibrations of an infinite viscoelastic layer with a dissipative memory. The Journal of the Acoustical Society of America 1974, 56 (3), 897-904.

[23] Cattaneo, C. Sulla conduzione del calore. Atti Sem. Mat. Fis. Univ. Modena 1948, 3, 83-101.

[24] Povstenko, Y. Fractional Cattaneo-type equations and generalized thermoelasticity. Journal of thermal Stresses 2011, 34 (2), 97-114.

[25] Povstenko, Y. Fractional Thermoelasticity; Springer International Publishing, 2015.

[26] Kalkal, K. K.; Deswal, S.; Yadav, R. Eigenvalue approach to fractional-order dual-phase-lag thermoviscoelastic problem of a thick plate. Iranian Journal of Science and Technology, Transactions of Mechanical Engineering 2019, 43, 917-927.

[27] Yang, W.; Chen, Z. Investigation of transient thermal-mechanical behavior of a cracked viscoelastic material using time-fractional dual-phase-lag theory. Theoretical and Applied Fracture Mechanics 2020, 106, 102500.

[28] Chakravorty, S.; Ghosh, S.; Sur, A. Thermo-viscoelastic interaction in a three-dimensional problem subjected to fractional heat conduction. Procedia engineering 2017, 173, 851-858.

[29] Pal, P.; Sur, A.; Kanoria, M. Thermo-viscoelastic interaction subjected to fractional Fourier law with three-phase-lag effects. Journal of Solid Mechanics 2015, 7 (4), 400-415.

[30] Youssef, H. M. Theory of fractional order generalized thermoelasticity. 2010.

[31] Hussein, E. M. Fractional model in the theory of generalized thermoelastic diffusion. Special Topics & Reviews in Porous Media: An International Journal 2023, 14 (1).

[32] Sherief, H. H.; Hussein, E. M. New fractional order model of thermoporoelastic theory for a porous infinitely long cylinder saturated with fluid. Waves in Random and Complex Media 2021, 1-30.

[33] Hussein, E. M. New fractional application on a homogenous isotropic thermo-poroelastic half-space. Mechanics of Solids 2022, 57 (4), 921-936.

[34] Adolfsson, K.; Enelund, M.; Olsson, P. On the fractional order model of viscoelasticity. Mechanics of Time-dependent materials 2005, 9, 15-34.
[35] Ezzat, M.; El-Karamany, A.; El-Bary, A. Thermo-viscoelastic materials with fractional relaxation operators. Applied Mathematical Modelling 2015, 39 (23-24), 7499-7512.
[36] Ezzat, M. A.; El-Karamany, A. S.; El-Bary, A. A.; Fayik, M. A. Fractional calculus in one-dimensional isotropic thermo-viscoelasticity. Comptes Rendus Mécanique 2013, 341 (7), 553-566.
[37] Hendy, M. H.; Amin, M. M.; Ezzat, M. A. Two-dimensional problem for thermoviscoelastic materials with fractional order heat transfer. Journal of Thermal Stresses 2019, 42 (10), 1298-1315.
[38] Li, C.; Guo, H.; Tian, X.; He, T. Generalized thermoviscoelastic analysis with fractional order strain in a thick viscoelastic plate of infinite extent. Journal of Thermal Stresses 2019, 42 (8), 1051-1070.
[39] Sherief, H. H.; Allam, M. N.; El-Hagary, M. A. Generalized theory of thermoviscoelasticity and a half-space problem. International Journal of Thermophysics 2011, 32, 1271-1295.
[40] Foutsitzi, G.; Kalpakidis, V.; Massalas, C. On the existence and uniqueness of linear thermoviscoelasticity zyxwvutsrqponm. Z. Angew. Math. Mech. 1997, 77 (1), 33-43.
[41] Fung, Y. Foundations of solid mechanics Prentice-Hall. Inc, New Jersey 1965.
[42] Ezzat, M. A.; El-Karamany, A. S.; El-Bary, A. A.; Fayik, M. A. Fractional calculus in one-dimensional isotropic thermo-viscoelasticity. Comptes rendus. Mécanique 2013, 341 (7), 553-566.
[43] Atangana, A.; Baleanu, D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. arXiv preprint arXiv:1602.03408 2016.
[44] Mainardi, F.; Gorenflo, R. Time-fractional derivatives in relaxation processes: a tutorial survey. arXiv preprint arXiv:0801.4914 2008.
[45] Hristov, J. The Craft of Fractional Modelling in Science and Engineering: II and III. MDPI: 2021; Vol. 5, p 281.
[46] Tarasov, V. E.; Tarasova, S. S. Fractional derivatives, and integrals: What are they needed for? Mathematics 2020, 8 (2), 164.
[47] Sherief, H. H.; Abd El-Latief, A. E. L. M.; Fayik, M. A. 2D hereditary thermoelastic application of a thick plate under axisymmetric temperature distribution. Mathematical Methods in the Applied Sciences 2022, 45 (2), 1080-1092.
[48] Sherief, H.; Abd El-Latief, A.; Fayik, M. The influence of an internal variable heat source on the perfect contact of three thermoelastic layers characterized by hereditary features. Mechanics of Time-Dependent Materials 2023, 1-21.
[49] Youssef, H. M.; El-Bary, A. A. The photothermal interaction of a viscothermoelastic semiconducting ceramic solid sphere under the Green–Naghdi heat conduction model. AIP Advances 2024, 14 (4).
[50] Zenkour, A.; Abouelregal, A. Effects of phase-lags in a thermoviscoelastic orthotropic continuum with a cylindrical hole and variable thermal conductivity. Archives of Mechanics 2015, 67 (6), 457-475.
[51] Honig, G.; Hirdes, U. A method for the numerical inversion of Laplace transforms. Journal of Computational and Applied Mathematics 1984, 10 (1), 113-132.
[52] Sherief, H.; Naim Anwar, M.; Abd El-Latief, A.; Fayik, M.; Tawfik, A. A fully coupled system of generalized thermoelastic theory for semiconductor medium. Scientific Reports 2024, 14 (1), 13876.
[53] Foutsitzi, G.; Kalpakidis, V.; Massalas, C. On the Existence and Uniqueness in Linear Thermoviscoelasticity zyxwvutsrqponm. ZAMM. Z. Angew. Math. Mech 1997, 77 (1), 33-43.
[54] Christensen, R. M.; Naghdi, P. M. Linear non-isothermal viscoelastic solids. Acta Mechanica 1967, 3 (1), 1-12.
[55] Awad, E. On the generalized thermal lagging behavior: Refined aspects. Journal of Thermal Stresses 2012, 35 (4), 293-325.
[56] Duffy, D. G. Transform methods for solving partial differential equations; Chapman and Hall/CRC, 2004.
[57] Sherief, H. H.; Hamza, F.; Abd El-Latief, A. 2D problem for a half-space in the generalized theory of thermo-viscoelasticity. Mechanics of Time-Dependent Materials 2015, 19, 557-568.
[58] Awad, E.; Fayik, M.; El-Dhaba, A. R. A comparative numerical study of a semi-infinite heat conductor subject to double strip heating under non-Fourier models. The European Physical Journal Plus 2022, 137 (12), 1303.
[59] Fayik, M.; Alhazmi, S. E.; Abdou, M. A.; Awad, E. Transient finite-speed heat transfer influence on deformation of a nanoplate with ultrafast circular ring heating. Mathematics 2023, 11 (5), 1099.
[60] Churchill, R. V. Operational mathematics. 1972.
[61] Boley, B. A. Discontinuities in integral-transform solutions. Quarterly of Applied Mathematics 1962, 19 (4), 273-284.
[62] Watson, G. N. A treatise on the theory of Bessel functions; The University Press, 1922.
[63] Chirita, S. Propagation of discontinuities in coupled micropolar thermoelasticity. Journal of Thermal Stresses 1980, 3 (2), 199-221.
[64] Fayik, M.; El Dhaba, A.; Awad, E. Diffusion-induced stresses due to an impulsive mass source under non-Fickian mass transfer models. 2022.
[65] Sherief, H. H.; Allam, M. N.; El-Hagary, M. A. Generalized theory of thermoviscoelasticity and a half-space problem. International Journal of Thermophysics 2011, 32 (6), 1271.
[66] Soygun, K.; Bolayir, G.; Boztug, A. Mechanical and thermal properties of polyamide versus reinforced PMMA denture base materials. The journal of advanced prosthodontics 2013, 5 (2), 153-160.

[67] Sherief, H. H.; Hamza, F. A. Generalized thermoelastic problem of a thick plate under axisymmetric temperature distribution. Journal of thermal stresses 1994, 17 (3), 435-452.

[68] Sherief, H. H.; Elmisiery, A.; Elhagary, M. Generalized thermoelastic problem for an infinitely long hollow cylinder for short times. Journal of Thermal Stresses 2004, 27 (10), 885-902.

[69] Fayik, M. Some boundary value problems in the theory of thermoelasticity. Mathematics, Applied Mathematics, Alexandria University, Faculty of Sciences, 2019.

[70] De Berg, M. Computational geometry: algorithms and applications; Springer Science & Business Media, 2000.

[71] Awad, E; Alhazmi, SE; Abdou, MA; Fayik, M. Anomalous Thermally Induced Deformation in Kelvin–Voigt Plate with Ultrafast Double-Strip Surface Heating. Fractal and Fractional. 2023,7(7), 563.

[72] Ezzat, M.; El-Karamany, A.; El-Bary, A. On thermo-viscoelasticity with variable thermal conductivity and fractional-order heat transfer. International Journal of Thermophysics 2015, 36 (7), 1684-1697.